

$\mathcal{W}_n^{(2)}$  ALGEBRAS

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ABSTRACT. We construct W-algebra generalizations of the  $\widehat{\mathfrak{sl}}(2)$  algebra — W algebras  $\mathcal{W}_n^{(2)}$  generated by two currents  $\mathcal{E}$  and  $\mathcal{F}$  with the highest pole of order  $n$  in their OPE. The  $n = 3$  term in this series is the Bershadsky–Polyakov  $W_3^{(2)}$  algebra. We define these algebras as a centralizer (commutant) of the  $\mathcal{U}_q \mathfrak{sl}(n|1)$  super quantum group and explicitly find the generators in a factored, “Miura-like” form. Another construction of the  $\mathcal{W}_n^{(2)}$  algebras is in terms of the coset  $\widehat{\mathfrak{sl}}(n|1)/\widehat{\mathfrak{sl}}(n)$ . The relation between the two constructions involves the “duality”  $(k+n-1)(k'+n-1) = 1$  between levels  $k$  and  $k'$  of two  $\widehat{\mathfrak{sl}}(n)$  algebras.

## 1. INTRODUCTION

The affine Lie algebra  $\widehat{\mathfrak{sl}}(2)$  (probably the second popular in conformal field theory after the incontestable Virasoro algebra) is not only the first term in the sequence  $\{\widehat{\mathfrak{sl}}(n)\}$  of affine Lie algebras, but also the first term in a sequence  $\{\mathcal{W}_n^{(2)}\}$  of W algebras generated by two dimension- $\frac{n}{2}$  currents  $\mathcal{E}_n(z)$  and  $\mathcal{F}_n(z)$  whose operator product expansion starts with a central term over the  $n$ th-order pole.

The  $\mathcal{W}_n^{(2)}$  algebras depend on a parameter  $k \in \mathbb{C}$ . For  $n = 2$ ,  $k$  is the  $\widehat{\mathfrak{sl}}(2)$  level; for  $n = 3$ ,  $\mathcal{W}_3^{(2)}$  is the Bershadsky–Polyakov (BP) W algebra  $W_3^{(2)}$  and  $k$  is the level of the  $\widehat{\mathfrak{sl}}(3)$  algebra from which it is obtained via the “defining” Hamiltonian reduction [1, 2, 3]. Each  $\mathcal{W}_n^{(2)}(k)$  contains a Virasoro subalgebra generated by the energy-momentum tensor  $\mathcal{T}_n(z)$  with the central charge

$$(1.1) \quad c_n(k) = -\frac{((k+n)(n-1)-n)((k+n)(n-2)n-n^2+1)}{k+n}$$

and a Heisenberg subalgebra generated by the modes of the dimension-1 current  $\mathcal{H}_n(z)$ , with the OPEs<sup>1</sup>

$$(1.2) \quad \begin{aligned} \mathcal{H}_n(z) \mathcal{H}_n(w) &= \frac{\ell_n(k)}{(z-w)^2}, & \ell_n(k) &= \frac{n-1}{n}k + n - 2, \\ \mathcal{H}_n(z) \mathcal{E}_n(w) &= \frac{\mathcal{E}_n}{z-w}, & \mathcal{H}_n(z) \mathcal{F}_n(w) &= -\frac{\mathcal{F}_n}{z-w}. \end{aligned}$$

<sup>1</sup>Normal-ordered products of composite operators are understood and regular terms are omitted in operator products.

The operator product expansion of  $\mathcal{E}_n$  and  $\mathcal{F}_n$  starts as

$$(1.3) \quad \mathcal{E}_n(z) \mathcal{F}_n(w) = \frac{\lambda_{n-1}(n, k)}{(z-w)^n} + \frac{n\lambda_{n-2}(n, k) \mathcal{H}_n(w)}{(z-w)^{n-1}} + \dots,$$

where  $\lambda_i(n, k)$  are numerical coefficients defined in (A.2) and the dots denote lower-order poles involving operators of dimensions  $\geq 2$ . The energy-momentum tensor is extracted from the pole of order  $n-2$  such that  $\mathcal{E}_n$  and  $\mathcal{F}_n$  are dimension- $\frac{n}{2}$  primary fields (and  $\mathcal{H}$  is a dimension-1 primary), see (A.1).

The notation  $\mathcal{W}_n^{(2)}$  extends the notation  $W_3^{(2)}$  used for the BP algebra, which was originally derived as a “second” Hamiltonian reduction of  $\widehat{sl}(3)$  and therefore had to be denoted similarly to but distinctly from  $W_3$ . We tend to interpret the superscript  $^{(2)}$  differently, as a reminder that the  $\mathcal{W}_n^{(2)}$  algebras are relatives of  $\widehat{sl}(2)$ . The lower  $\mathcal{W}_n^{(2)}$  algebras — the Bershadsky–Polyakov algebra  $W_3^{(2)}$  and the otherwise not celebrated algebra  $W_4^{(2)}$  — are described in more detail in Appendix A.

Each algebra  $\mathcal{W}_n^{(2)}$  can be derived in at least two different ways: from the centralizer of the quantum group  $\mathcal{U}_q sl(n|1)$  and from the coset theory  $\widehat{sl}(n|1)/\widehat{sl}(n)$ .

**$\mathcal{W}_n^{(2)}$  as a centralizer of  $\mathcal{U}_q sl(n|1)$ .** The standard definition of the W algebra associated with a given root system is as the centralizer of the corresponding quantum group [4] (cf. [5]). We recall that a general principle in the theory of vertex-operator algebras consists in a certain duality between vertex-operator algebras and quantum groups. In free-field realizations of a vertex-operator algebra  $\mathcal{A}$ , the corresponding quantum group  $\mathcal{U}_q \mathfrak{g}$  is represented by screening operators; more precisely, screenings are elements of  $\mathcal{U}_q \mathfrak{n}$ , the nilpotent subalgebra in  $\mathcal{U}_q \mathfrak{g}$ . They can be thought of as symmetries of the conformal field theory associated with  $\mathcal{A}$ : the two algebras,  $\mathcal{A}$  and  $\mathcal{U}_q \mathfrak{g}$ , are each other’s centralizers (commutants) in the algebra of free fields. More precisely, the vertex-operator algebra  $\mathcal{A}$  is the centralizer of  $\mathcal{U}_q \mathfrak{g}$  in the algebra of *local* operators; in turn, the centralizer of  $\mathcal{A}$  in fact contains *two* quantum groups, which commute with each other and *one* of which typically suffices to single out  $\mathcal{A}$  as its centralizer.<sup>2</sup>

In the standard construction of W algebras, the centralizer of  $\mathcal{U}_q \mathfrak{g}$  is sought in the algebra  $\mathcal{V}$  of operators that are descendants of the identity (in other words, correspond to states in the vacuum representation). We generalize this by first adding one extra Heisenberg algebra (scalar current) and then *extending  $\mathcal{V}$  by a one-dimensional lattice vertex-operator algebra*. We thus seek the quantum group centralizer in  $\mathcal{V}_\xi = \{\mathcal{P}(\partial\varphi) e^{m(\xi, \varphi)} \mid m \in \mathbb{Z}\}$ , where  $\xi$  is a lightlike lattice vector,  $\varphi$  is a collection of scalar fields,  $(\ , \ )$  is the Euclidean

<sup>2</sup>For example, for Virasoro theories, both quantum groups are the quantum  $sl(2)$ ; for  $\widehat{sl}(2)$  theories, the two quantum groups are  $\mathcal{U}_q sl(2|1)$  and  $\mathcal{U}_{q'} sl(2)$ ; for the coset  $\widehat{sl}(2) \oplus \widehat{sl}(2)/\widehat{sl}(2)$ , these are  $\mathcal{U}_q D(2; 1|\alpha)$  and  $\mathcal{U}_{q_1} sl(2) \otimes \mathcal{U}_{q_2} sl(2) \otimes \mathcal{U}_{q_3} sl(2)$  [6].

scalar product, and  $\mathcal{P}$  are differential polynomials in all components of  $\partial\varphi$  (and  $\partial = \frac{\partial}{\partial z}$ ); this involves one scalar field (Heisenberg algebra) more than in the standard construction. For  $\xi$  chosen specially (and for generic values of  $k$ ), such a centralizer of  $\mathcal{U}_q s\ell(n|1)$  is the  $\mathcal{W}_n^{(2)}$  algebra; the required choice of the vector  $\xi$  is determined by quantum group considerations, as we discuss in detail in the paper.

We find the  $\mathcal{W}_n^{(2)}$  generators  $(\mathcal{E}_n(z), \mathcal{H}_n(z), \mathcal{F}_n(z))$  in the centralizer of  $\mathcal{U}_q s\ell(n|1)$  explicitly. Before describing our construction, we note that because  $s\ell(n|1)$  admits inequivalent simple root systems, we have to consider different realizations  $\mathcal{W}_{n[m]}^{(2)}$ ,  $0 \leq m \leq n$ , which are isomorphic W algebras, but whose generators are constructed through free fields differently.<sup>3</sup> For  $m = 0$  and  $m = n$ , the respective Dynkin diagrams are (with  $n$  nodes in each case)

$$\bullet \text{---} \bullet \text{---} \dots \text{---} \bullet \text{---} \circ$$

and

$$\circ \text{---} \bullet \text{---} \dots \text{---} \bullet \text{---} \bullet$$

which are the same. For  $1 \leq m \leq n-1$ , the corresponding Dynkin diagrams are ( $n$  nodes)

$$\bullet \text{---} \bullet \text{---} \dots \text{---} \bullet \text{---} \circ \text{---} \circ \text{---} \bullet \text{---} \dots \text{---} \bullet$$

with  $n-m-1$  black dots (even roots) to the left of the white dots (odd roots). To describe the generators in the centralizer of the corresponding quantum group, we let  $R_i^+$ ,  $i = 0, \dots, n-m-1$ , and  $R_i^-$ ,  $i = 0, \dots, m-1$ , be free-field currents with the OPEs

$$(1.4) \quad \begin{aligned} R_i^+(z) R_i^+(w) &= R_i^-(z) R_i^-(w) = \frac{1}{(z-w)^2}, \\ R_i^+(z) R_j^+(w) &= R_i^-(z) R_j^-(w) = \frac{-k-n+1}{(z-w)^2}, \quad i \neq j, \\ R_i^+(z) R_j^-(w) &= \frac{k+n-1}{(z-w)^2}, \end{aligned}$$

and let  $Y$  be the current with the OPEs

$$(1.5) \quad R_i^\pm(z) Y(w) = \frac{\pm 1}{(z-w)^2}, \quad Y(z) Y(w) = 0$$

and  $\Xi$  the corresponding scalar field, such that  $\partial e^{\Xi(z)} = :Y e^{\Xi}: (z)$  (with  $\partial = \frac{\partial}{\partial z}$ ).

**1.1. Theorem.** *The two currents generating  $\mathcal{W}_{n[m]}^{(2)}(k)$  are given by*

$$\begin{aligned} \mathcal{E}_{n[m]}(z) &= : \left( ((k+n-1)\partial + R_{m-1}^-(z)) \dots ((k+n-1)\partial + R_1^-(z)) R_0^-(z) \right) e^{\Xi(z)} :, \\ \mathcal{F}_{n[m]}(z) &= : \left( ((k+n-1)\partial + R_{n-m-1}^+(z)) \dots ((k+n-1)\partial + R_1^+(z)) R_0^+(z) \right) e^{-\Xi(z)} :. \end{aligned}$$

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<sup>3</sup>In general, there seems to be no *theorem* that the W algebras obtained as centralizers of screenings corresponding to different root systems of the same algebra are isomorphic, but in all known examples, the different root systems lead to the same algebra.

Here,  $\partial = \frac{\partial}{\partial z}$  and the action of the derivatives is delimited by the outer brackets (i.e., the derivatives do not act on the exponentials). For  $m = 1$ , only the  $R_0^-$  factor is involved in  $\mathcal{E}_{n[1]}$ , and for  $m = 0$ ,  $\mathcal{E}_{n[0]}(z) = e^{\Xi(z)}$ . For  $m = n - 1$ , only the  $R_0^+$  factor is involved in  $\mathcal{F}_{n[n-1]}$ , and for  $m = n$ ,  $\mathcal{F}_{n[n]}(z) = e^{-\Xi(z)}$ .

These formulas can be obtained by noting that free-field realizations of the generators of  $\mathcal{W}_{n[m]}^{(2)}(k)$  are related to free-field realizations of the generators of  $\mathcal{W}_{(n-1)[m]}^{(2)}(k+1)$ . They generalize the following well-known situation: the bosonic  $\beta\gamma$  system (which is  $\mathcal{W}_1^{(2)}$ ) is bosonized through two scalar fields; this bosonization is involved in the symmetric realization of  $\widehat{sl}(2)$  (which is  $\mathcal{W}_{2[1]}^{(2)}$ ), which can be obtained by “rebosonizing” the Wakimoto representation. This has an analogue for all  $n$ ; in addition, relations between the maximally asymmetric realizations of two subsequent algebras lead to the factored form of the generators in Theorem 1.1. We do not prove that the  $\mathcal{E}_{n[m]}(z)$  and  $\mathcal{F}_{n[m]}(z)$  currents in Theorem 1.1 generate the entire centralizer, but we believe that this is true for generic  $k$ .

It follows from the definition that  $\mathcal{W}_n^{(2)}$  contains the subalgebra  $\mathcal{W}sl(n|1) \otimes \widehat{\mathcal{U}\mathfrak{h}}$ , where  $\mathcal{W}sl(n|1)$  is the standard W algebra associated with the  $sl(n|1)$  root system and  $\widehat{\mathfrak{h}}$  is a Heisenberg algebra commuting with it. The  $\mathcal{W}sl(n|1) \otimes \widehat{\mathcal{U}\mathfrak{h}}$  subalgebra is merely the “zero momentum” sector of  $\mathcal{W}_n^{(2)}$ , i.e., consists of elements in the centralizer of  $\mathcal{U}_qsl(n|1)$  in  $\mathcal{V}$ , descendants of the identity operator. Introducing the generalized parafermions  $\overline{\mathcal{W}}_n^{(2)}$  as the quotient over the Heisenberg subalgebra, we thus have

$$\overline{\mathcal{W}}_n^{(2)} = \mathcal{W}_n^{(2)} / \widehat{\mathfrak{h}}, \quad \overline{\mathcal{W}}_n^{(2)} \supset \mathcal{W}sl(n|1).$$

The algebra  $\overline{\mathcal{W}}_n^{(2)}$  is nonlocal, but we use it because it naturally appears in some constructions and its locality can easily be “restored” by tensoring with an additional free scalar field, thus recovering  $\mathcal{W}_n^{(2)}$ .

Although this is not in the focus of our attention in this paper, we note that  $\mathcal{W}sl(n|1)$  are the “unifying W algebras” [7] that interpolate the rank of  $\mathcal{W}sl(\cdot)$  algebras. The underlying “numerology” is as follows [7]. We take the W algebra  $\mathcal{W}sl(m)$ ,  $m \in \mathbb{N}$ , and consider its  $(p, p')$  minimal model. Its central charge is given by

$$c_{p,p'}(m) = 2m^3 - m - 1 - (m-1)m(m+1)\frac{p'}{p} - (m-1)m(m+1)\frac{p}{p'}.$$

The central charge of  $\mathcal{W}_n^{(2)}(k)$  with  $k = 1 - n + \frac{m+1}{n-1}$  satisfies the relation

$$c_n(k) - 1 = c_{m+1, m+n}(m).$$

This suggests a rank–level duality of the corresponding minimal models,

$$(1.6) \quad \mathcal{W}sl(n|1)_{1-n+\frac{m+1}{n-1}} = \mathcal{W}sl(m)_{m+n, m+1}.$$

The algebra of this minimal model closes on normal-ordered differential polynomials in the currents of conformal dimensions  $2, 3, \dots, 2n+1$  [7], which is often expressed in the notation  $\mathcal{W}(2, 3, \dots, 2n+1)$  for the model. The mechanism underlying (1.6) was also discussed in [8]. The  $\mathcal{W}sl(m)_{m+n, m+1}$  minimal model can be extended by taking the tensor product with a certain lattice vertex-operator algebra generated by a free field  $\varphi$ . A certain primary field of  $\mathcal{W}sl(m)_{m+n, m+1}$  can then be “dressed” with  $e^{\alpha\varphi}$  such that the resulting operator becomes local and can be identified with the  $\mathcal{E}$  current of  $\mathcal{W}_n^{(2)}$ ; another (“dual”) primary is dressed into the  $\mathcal{F}$  current by  $e^{-\alpha\varphi}$ . This gives “integrable” representations of  $\mathcal{W}_n^{(2)}$ , generalizing the integrable representations of  $\widehat{sl}(2)$ .

**$\mathcal{W}_n^{(2)}$  algebras from  $\widehat{sl}(n|1)$ .** Alternatively to the quantum-group description, the  $\mathcal{W}_n^{(2)}(k)$  algebras can be given another characterization, in terms of coset conformal field theories. The algebras involved in the coset construction are with the “dual” level  $k'$  related to  $k$  by

$$(1.7) \quad (k+n-1)(k'+n-1) = 1.$$

For  $k \neq -n$ , Eq. (1.7) is equivalent to

$$(1.8) \quad \frac{1}{k+n} + \frac{1}{k'+n} = 1.$$

**1.2. Theorem.** *Let  $k'$  be related to  $k$  by (1.7). Then the coset of  $\widehat{sl}(n|1)_{k'}$  with respect to its even subalgebra  $\widehat{gl}(n)_{k'}$  is given by*

$$(1.9) \quad \widehat{sl}(n|1)_{k'}/\widehat{gl}(n)_{k'} = \mathcal{W}_{n[m]}^{(2)}(k)/\widehat{\mathfrak{h}}$$

(where the  $\widehat{\mathfrak{h}}$  subalgebra is generated by (the modes of) the current  $\mathcal{H}_n(z)$ ) for any  $m \in [0, \dots, n]$ . In particular, all the  $\mathcal{W}_{n[m]}^{(2)}$  algebras with  $m \in [0, n]$  are isomorphic.

For  $n = 2$ , we thus recover the well-known identification  $\widehat{sl}(2)/\widehat{\mathfrak{h}} = \widehat{sl}(2|1)/\widehat{gl}(2)$ .

The cosets  $\widehat{sl}(n|1)/\widehat{sl}(n)$ , which are “almost” the left-hand side of (1.9), can be constructed in rather explicit terms as follows. In  $\widehat{sl}(n|1)$ , the  $2n$  fermions are organized into two  $sl(n)$   $n$ -plets  $\mathbb{C}^n(z)$  and  $\mathbb{C}^n(z)$ . Then  $\bigwedge^n \mathbb{C}^n(z)$  and  $\bigwedge^n \mathbb{C}^n(z)$  are in the centralizer of the  $\widehat{sl}(n)$  subalgebra. The coset  $\widehat{sl}(n|1)/\widehat{sl}(n)$  is isomorphic to the centralizer of  $\widehat{sl}(n)$  in  $\mathcal{U}\widehat{sl}(n|1)$ . This is an essential simplification compared with the general case, where a coset is usually defined as the cohomology of some BRST operator, and only the Virasoro algebra can be realized by operators commuting with the subalgebra. The cosets  $\widehat{sl}(n|1)/\widehat{sl}(n)$  is quite close to the  $\mathcal{W}_n^{(2)}$  algebra: after a “correction” by  $e^{\pm\sqrt{n}\phi(z)}$ , where  $\phi$  is an auxiliary scalar with the OPE

$$(1.10) \quad \partial\phi(z)\partial\phi(w) = -\frac{1}{(z-w)^2},$$

$\bigwedge^n \mathbb{C}^n(z)$  and  $\bigwedge^n \mathbb{C}^n(z)$  become the two currents generating  $\mathcal{W}_n^{(2)}(k)$ .

The contents of this paper can be outlined as follows. In Sec. 2, we study the  $\mathcal{W}_n^{(2)}$  algebra defined as the centralizer of  $\mathcal{U}_q s\ell(n|1)$  in a certain lattice vertex-operator algebra. In 2.1, we define the different *realizations* of  $\mathcal{W}_n^{(2)}$ , denoted by  $\mathcal{W}_{n[m]}^{(2)}$ . In 2.2, we describe the motivation of our approach leading to the construction of the  $\mathcal{W}_{n[m]}^{(2)}$  generators in the centralizer of the corresponding screenings. Actual calculations, eventually leading to Theorem 1.1, are given in 2.3. The OPEs following from Theorem 1.1 are considered in 2.4. In Sec. 3, we alternatively define  $\mathcal{W}_n^{(2)}$  in terms of the coset  $\widehat{s\ell}(n|1)/\widehat{s\ell}(n)$ . The actual statement to be proved is in Theorem 3.1 and the proof is outlined in 3.2. It involves finding another set of screenings representing the  $\mathcal{U}_{q'} s\ell(n)$  quantum group. The quantum-group structure is then used to construct a vertex-operator extension of  $\mathcal{W}_n^{(2)}$  by the “denominator”  $\widehat{g\ell}(n)$  into the “numerator”  $\widehat{s\ell}(n|1)$ , thus inverting the coset and hence showing that  $\mathcal{W}_n^{(2)}$  is indeed given by the coset construction. Some speculations on vertex-operator extensions are given in 3.3.

## 2. $\mathcal{W}_n^{(2)}$ ALGEBRAS FROM $\mathcal{U}_q s\ell(n|1)$

In this section, we define the  $\mathcal{W}_n^{(2)}$  algebra as the centralizer of  $\mathcal{U}_q s\ell(n|1)$  in a certain lattice vertex-operator algebra. We then find the algebra generators in the centralizer; we do not prove that the centralizer is thus exhausted, but we believe that it is for generic  $k$ .

**2.1. Realizations.** To define the  $\mathcal{W}_n^{(2)}$  algebra as a centralizer of the screenings representing (the nilpotent subalgebra of)  $\mathcal{U}_q s\ell(n|1)$ , we proceed as follows. We represent  $\mathfrak{n}_q s\ell(n|1)$ , the nilpotent subalgebra of  $\mathcal{U}_q s\ell(n|1)$ , by operators expressed through  $n+1$  scalar fields  $\varphi_i$ ,  $i = 1, \dots, n+1$ . These operators are called screenings in what follows. We next extend the space of differential polynomials in the currents  $\partial\varphi_1, \dots, \partial\varphi_{n+1}$  by adding a one-dimensional lattice vertex-operator algebra, i.e., the operators  $e^{m(\xi, \varphi)}$ ,  $m \in \mathbb{Z}$ , with a specially chosen vector  $\xi$ . The  $\mathcal{W}_n^{(2)}$  algebra is then defined as the centralizer of the  $\mathfrak{n}_q s\ell(n|1)$ -screenings in the space  $\mathcal{V}_\xi = \{\mathcal{P}(\partial\varphi) e^{m(\xi, \varphi)} \mid m \in \mathbb{Z}\}$ , where  $\mathcal{P}$  are differential polynomials.

To specify this in more detail, we note that the  $s\ell(n|1)$  Lie *superalgebra* admits inequivalent simple root systems, and we must therefore distinguish between centralizers of the screenings associated with each of the inequivalent simple root systems. To the algebras constructed for each of the root systems, we refer as *realizations* of  $\mathcal{W}_n^{(2)}$ , to be denoted by  $\mathcal{W}_{n[m]}^{(2)}$ . We now consider the definitions of  $\mathcal{W}_{n[m]}^{(2)}$ .

**2.1.1.  $n[0]$ .** The *maximally asymmetric* realization, denoted by  $\mathcal{W}_{n[0]}^{(2)}$ , corresponds to the simple root system of  $s\ell(n|1)$  represented by the Dynkin diagram

$$\bullet \text{---} \bullet \text{---} \bullet \text{---} \dots \text{---} \bullet \text{---} \circ,$$

where filled (open) dots denote even (odd) roots. The corresponding Cartan matrix of  $sl(n|1)$  is given by

$$\begin{pmatrix} 2 & -1 & 0 & \dots & \dots & 0 \\ -1 & 2 & -1 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & -1 & 2 & -1 \\ 0 & \dots & 0 & -1 & 0 & 0 \end{pmatrix}.$$

From this Cartan matrix, we construct the screenings representing  $\mathfrak{n}_q sl(n|1)$ . For this, we first introduce vectors  $\mathbf{a}_{n-1}, \dots, \mathbf{a}_1, \boldsymbol{\psi}$ , in  $\mathbb{C}^n$  whose Gram matrix (pairwise scalar products) is given by the “dressed Cartan matrix”

$$(2.1) \quad \begin{array}{c} \mathbf{a}_{n-1} \\ \mathbf{a}_{n-2} \\ \dots \\ \mathbf{a}_1 \\ \boldsymbol{\psi} \end{array} \left| \begin{array}{cccccc} 2(k+n) & -k-n & 0 & \dots & \dots & 0 \\ -k-n & 2(k+n) & -k-n & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & -k-n & 2(k+n) & -k-n \\ 0 & \dots & 0 & 0 & -k-n & 1 \end{array} \right.$$

where the leftmost column indicates labeling of rows. The determinant of this matrix is  $-(k+n)^{n-1} n \ell_n(k)$ , and the vectors are therefore determined uniquely modulo a common rotation for  $k \in \mathbb{C} \setminus \{-n, -\frac{n(n-2)}{n-1}\}$ .

Let  $\varphi$  be the  $n$ -plet of scalar fields with the OPEs

$$\partial \varphi_i(z) \partial \varphi_j(w) = \frac{\delta_{i,j}}{(z-w)^2}.$$

With the above vectors  $\mathbf{a}_{n-1}, \dots, \mathbf{a}_1, \boldsymbol{\psi}$ , we define the operators

$$E_i = \oint e^{\mathbf{a}_i \cdot \boldsymbol{\varphi}}, \quad i = 1, \dots, n-1, \quad \Psi = \oint e^{\boldsymbol{\psi} \cdot \boldsymbol{\varphi}},$$

where the dot denotes the Euclidean scalar product in  $\mathbb{C}^n$ . These *screening operators* represent  $\mathfrak{n}_q sl(n|1)$ . The screenings  $E_i$  are said to be *bosonic* and  $\Psi$  *fermionic*.

We next define the vector  $\boldsymbol{\xi} \in \mathbb{C}^n$  by its scalar products with  $\mathbf{a}_1, \dots, \mathbf{a}_{n-1}, \boldsymbol{\psi}$ ,

$$(2.2) \quad \begin{aligned} \boldsymbol{\xi} \cdot \mathbf{a}_i &= 0, \quad i = 1, \dots, n-1, \\ \boldsymbol{\xi} \cdot \boldsymbol{\psi} &= 1 \end{aligned}$$

and set  $\overline{\mathcal{V}}_{\boldsymbol{\xi}} = \{\mathcal{P}(\partial \boldsymbol{\varphi}) e^{m \boldsymbol{\xi} \cdot \boldsymbol{\varphi}} \mid m \in \mathbb{Z}\}$ . By definition, the  $n[0]$  realization of  $\overline{\mathcal{W}}_n^{(2)}$ , denoted by  $\overline{\mathcal{W}}_{n[0]}^{(2)}$ , is the centralizer of  $(E_i)_{i=1, \dots, n-1}$  and  $\Psi$  in  $\overline{\mathcal{V}}_{\boldsymbol{\xi}}$ . Restoring locality — i.e., reconstructing  $\mathcal{W}_{n[0]}^{(2)}$  — requires one scalar field more.

To construct  $\mathcal{W}_n^{(2)}$ , we embed  $\mathbb{C}^n$  in  $\mathbb{C}^{n+1}$  as a coordinate hyperplane, let  $\psi, a_1, \dots, a_{n-1}$  denote the respective images of  $\boldsymbol{\psi}, \mathbf{a}_1, \dots, \mathbf{a}_{n-1}$ , and extend the set of free scalar

fields accordingly, to  $\partial\varphi = \{\partial\varphi, \partial\varphi_{n+1}\}$ . In addition, we define  $\xi = \{\xi, \xi_{n+1}\}$  such that  $\xi$  is isotropic with respect to the Euclidean scalar product  $(\cdot, \cdot)$  in  $\mathbb{C}^{n+1}$ :

$$(\xi, \xi) = \xi \cdot \xi + \xi_{n+1}^2 = 0$$

(the other scalar products remain the same as among the  $n$ -dimensional vectors above).

With a slight abuse of notation, we then let  $\Psi$  and  $E_i$  denote the screening operators

$$(2.3) \quad E_i = \oint e^{(a_i, \varphi)}, \quad i = 1, \dots, n-1, \quad \Psi = \oint e^{(\psi, \varphi)}.$$

The  $n[0]$  realization of  $\mathcal{W}_n^{(2)}$ , denoted by  $\mathcal{W}_{n[0]}^{(2)}$ , is the centralizer of  $(E_i)_{i=1, \dots, n-1}$ , and  $\Psi$  in  $\mathcal{V}_\xi = \{\mathcal{P}(\partial\varphi) e^{m(\xi, \varphi)} \mid m \in \mathbb{Z}\}$ .

In what follows, the differential polynomials in  $\partial\varphi$  are expressed through the currents

$$A_i = (a_i, \partial\varphi), \quad Q = (\psi, \partial\varphi), \quad Y = (\xi, \partial\varphi),$$

which have the nonzero operator products (read off from the matrix (2.1))

$$\begin{aligned} A_i(z) A_{i+1}(w) &= \frac{-k-n}{(z-w)^2}, \quad A_i(z) A_i(w) = \frac{2(k+n)}{(z-w)^2}, \\ A_1(z) Q(w) &= \frac{-k-n}{(z-w)^2}, \quad Q(z) Q(w) = \frac{1}{(z-w)^2}, \quad Q(z) Y(w) = \frac{1}{(z-w)^2}. \end{aligned}$$

**2.1.2. Remark.** It follows from (2.2) that the  $n$ -dimensional part  $\xi$  of  $\xi$  is given by

$$(2.4) \quad \xi = -\frac{1}{n\ell_n(k)} (a_{n-1} + 2a_{n-2} + \dots + (n-1)a_1 + n\psi)$$

and

$$\xi \cdot \xi = -\frac{1}{\ell_n(k)}.$$

We also note that the determinant of the  $(n+1) \times (n+1)$  Gram matrix of the vectors  $a_{n-1}, \dots, a_1, \psi, \xi$ ,

$$(2.5) \quad \Gamma_n(k) = \begin{pmatrix} 2(k+n) & -k-n & 0 & \dots & 0 & 0 \\ -k-n & 2(k+n) & -k-n & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & -k-n & 2(k+n) & -k-n & 0 \\ 0 & \dots & 0 & 0 & -k-n & 1 & 1 \\ 0 & \dots & 0 & 0 & 0 & 1 & 0 \end{pmatrix},$$

is given by  $-n(k+n)^{n-1}$ , and hence these  $n+1$  vectors form a basis in  $\mathbb{C}^{n+1}$  and are determined uniquely modulo a common rotation for  $k \in \mathbb{C} \setminus \{-n\}$ . For future use, we



note that

$$\Gamma_n(k)^{-1} = \frac{1}{n} \times \begin{pmatrix} \frac{(n-1) \cdot 1}{k+n} & \frac{(n-2) \cdot 1}{k+n} & \frac{(n-3) \cdot 1}{k+n} & \dots & \dots & \dots & \frac{1 \cdot 1}{k+n} & 0 & 1 \\ \frac{(n-2) \cdot 1}{k+n} & \frac{(n-2) \cdot 2}{k+n} & \frac{(n-3) \cdot 2}{k+n} & \frac{(n-4) \cdot 2}{k+n} & \dots & \dots & \frac{1 \cdot 2}{k+n} & 0 & 2 \\ \frac{(n-3) \cdot 1}{k+n} & \frac{(n-3) \cdot 2}{k+n} & \frac{(n-3) \cdot 3}{k+n} & \frac{(n-4) \cdot 3}{k+n} & \frac{(n-5) \cdot 3}{k+n} & \dots & \frac{1 \cdot 3}{k+n} & 0 & 3 \\ \frac{(n-4) \cdot 1}{k+n} & \frac{(n-4) \cdot 2}{k+n} & \frac{(n-4) \cdot 3}{k+n} & \frac{(n-4) \cdot 4}{k+n} & \frac{(n-5) \cdot 4}{k+n} & \frac{(n-6) \cdot 4}{k+n} & \dots & \frac{1 \cdot 4}{k+n} & 0 & 4 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \frac{1 \cdot 1}{k+n} & \frac{1 \cdot 2}{k+n} & \frac{1 \cdot 3}{k+n} & \frac{1 \cdot 4}{k+n} & \dots & \dots & \frac{1 \cdot (n-1)}{k+n} & 0 & n-1 \\ 0 & 0 & \dots & \dots & \dots & \dots & 0 & 0 & n \\ 1 & 2 & 3 & 4 & \dots & \dots & n-1 & n & n\ell_n(k) \end{pmatrix}.$$

**2.1.3.  $n[1]$ .** The realization  $\mathcal{W}_{n[1]}^{(2)}$  corresponds to the Dynkin diagram



with two odd roots. Similarly to the  $n[0]$  case, we introduce vectors  $\mathbf{a}_{n-2}, \dots, \mathbf{a}_1, \psi_+, \psi_-$  in  $\mathbb{C}^n$  whose Gram matrix is given by the Cartan matrix “dressed” into

$$\begin{pmatrix} \mathbf{a}_{n-2} \\ \mathbf{a}_{n-3} \\ \dots \\ \mathbf{a}_1 \\ \psi_+ \\ \psi_- \end{pmatrix} \begin{pmatrix} 2(k+n) & -k-n & 0 & \dots & \dots & 0 \\ -k-n & 2(k+n) & -k-n & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & -k-n & 2(k+n) & -k-n & 0 \\ 0 & \dots & \dots & 0 & -k-n & 1 & k+n-1 \\ 0 & \dots & \dots & \dots & 0 & k+n-1 & 1 \end{pmatrix}.$$

We also introduce an  $n$ -tuple of scalar fields  $\varphi$  and define the screenings

$$E_i = \oint e^{\mathbf{a}_i \cdot \varphi}, \quad i = 1, \dots, n-2, \quad \Psi_+ = \oint e^{\psi_+ \cdot \varphi}, \quad \Psi_- = \oint e^{\psi_- \cdot \varphi},$$

where the dot denotes the Euclidean scalar product in  $\mathbb{C}^n$ . These screenings represent  $n_q s\ell(n|1)$ . We next define the vector  $\xi$  by its products with  $\mathbf{a}_{n-2}, \dots, \mathbf{a}_1, \psi_+, \psi_-$ ,

$$\begin{aligned} \xi \cdot \mathbf{a}_i &= 0, \quad i = 1, \dots, n-2, \\ \xi \cdot \psi_+ &= 1, \\ \xi \cdot \psi_- &= -1. \end{aligned} \tag{2.6}$$

The  $n[1]$  realization of  $\overline{\mathcal{W}}_n^{(2)}$ , denoted by  $\overline{\mathcal{W}}_{n[1]}^{(2)}$ , is the centralizer of  $(E_i)_{i=1, \dots, n-2}, \Psi_+, \Psi_-$  in  $\overline{\mathcal{V}}_\xi = \{\mathcal{P}(\partial\varphi) e^{m\xi \cdot \varphi} \mid m \in \mathbb{Z}\}$ .

To construct  $\mathcal{W}_{n[1]}^{(2)}$ , we embed  $\mathbb{C}^n$  in  $\mathbb{C}^{n+1}$  as the coordinate hyperplane, let  $a_{n-2}, \dots, a_1, \psi_+, \psi_-$  denote the respective images of  $\mathbf{a}_{n-2}, \dots, \mathbf{a}_1, \boldsymbol{\psi}_+, \boldsymbol{\psi}_-$ , and extend the free scalar fields as  $\partial\varphi = \{\partial\varphi, \partial\varphi_{n+1}\}$ . In addition, we define  $\xi = \{\xi, \xi_{n+1}\}$  such that  $\xi$  is isotropic with respect to the Euclidean scalar product  $(\cdot, \cdot)$  in  $\mathbb{C}^{n+1}$  (the other scalar products remain the same as among the  $n$ -dimensional vectors above). We again use  $\Psi_-, \Psi_+$ , and  $E_i$  to denote the screenings

$$E_i = \oint e^{(a_i, \varphi)}, \quad i = 1, \dots, n-2, \quad \Psi_+ = \oint e^{(\psi_+, \varphi)}, \quad \Psi_- = \oint e^{(\psi_-, \varphi)}.$$

The screenings  $\Psi_+$  and  $\Psi_-$  are said to be *fermionic* and the other screenings *bosonic*.

The  $n[1]$  realization  $\mathcal{W}_{n[1]}^{(2)}$  of  $\mathcal{W}_n^{(2)}$  is the centralizer of  $(E_i)_{i=1, \dots, n-2}, \Psi_+, \Psi_-$  in  $\mathcal{V}_\xi = \{\mathcal{P}(\partial\varphi) e^{m(\xi, \varphi)} \mid m \in \mathbb{Z}\}$ . As in the maximally asymmetric case, we express differential polynomials in  $\partial\varphi$  through the currents

$$A_i = (a_i, \partial\varphi), \quad Q_\pm = (\psi_\pm, \partial\varphi), \quad Y = (\xi, \partial\varphi),$$

whose OPEs are determined by the matrix above and scalar products (2.6),

$$\begin{aligned} A_i(z) A_{i+1}(w) &= \frac{-k-n}{(z-w)^2}, & A_i(z) A_i(w) &= \frac{2(k+n)}{(z-w)^2}, \\ A_1(z) Q_+(w) &= \frac{-k-n}{(z-w)^2}, & Q_+(z) Q_-(w) &= \frac{k+n-1}{(z-w)^2}, \\ Q_\pm(z) Q_\pm(w) &= \frac{1}{(z-w)^2}, & Q_\pm(z) Y(w) &= \frac{\pm 1}{(z-w)^2}. \end{aligned}$$

**2.1.4. Remark.** It follows that

$$\boldsymbol{\xi} = -\frac{1}{n\ell_n(k)}(\mathbf{a}_{n-2} + 2\mathbf{a}_{n-3} + \dots + (n-2)\mathbf{a}_1 + (n-1)\boldsymbol{\psi}_+ - \boldsymbol{\psi}_-)$$

and

$$\boldsymbol{\xi} \cdot \boldsymbol{\xi} = -\frac{1}{\ell_n(k)}.$$

The determinant of the  $(n+1) \times (n+1)$  Gram matrix of the vectors  $a_{n-2}, \dots, a_1, \psi_+, \psi_-, \xi$  is given by  $-n(k+n)^{n-1}$ , and these  $n+1$  vectors therefore form a basis in  $\mathbb{C}^{n+1}$  and are determined uniquely modulo a common rotation for all  $k \in \mathbb{C} \setminus \{-n\}$ .

**2.1.5.  $n[m]$ .** The subsequent  $n[m]$  realizations correspond to the Dynkin diagrams

$$\bullet \text{---} \bullet \text{---} \dots \text{---} \bullet \text{---} \circ \text{---} \circ \text{---} \bullet \text{---} \dots \text{---} \bullet,$$

with  $m-1$  even roots to the right of the odd roots. We can restrict  $m$  to  $0 \leq m \leq [\frac{n}{2}]$ , because taking  $m > [\frac{n}{2}]$  amounts to applying an automorphism (inducing  $\mathcal{E} \leftrightarrow \mathcal{F}$  on the algebra) to the realization where  $m$  is replaced by  $n-m$ .

We do not repeat the definition of  $\overline{\mathcal{W}}_{n[m]}^{(2)}$  in terms of  $n$  scalar fields and proceed to defining  $\mathcal{W}_{n[m]}^{(2)}$  in terms of  $n+1$  scalar fields. For this, we introduce  $(n+1)$ -dimensional

vectors  $a_{n-m-1}, \dots, a_1, \psi_+, \psi_-, a_{-1}, \dots, a_{-m+1}, \xi$  whose Gram matrix (with the determinant  $-n(k+n)^{n-1}$ ) is given by

$$\begin{array}{c} a_{n-m-1} \\ a_{n-m-2} \\ \vdots \\ a_1 \\ \psi_+ \\ \psi_- \\ a_{-1} \\ \vdots \\ a_{-2} \\ a_{-m+1} \\ \xi \end{array} \left| \begin{array}{cccccccccccc} \underline{2K} & -K & 0 & \dots & \dots & \dots & 0 & 0 \\ -K & \underline{2K} & -K & 0 & \dots & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & -K & \underline{2K} & -K & 0 & \dots & 0 & 0 \\ 0 & \dots & 0 & -K & \underline{1} & K-1 & 0 & \dots & 0 & 1 \\ 0 & \dots & 0 & K-1 & \underline{1} & -K & 0 & \dots & 0 & -1 \\ 0 & \dots & 0 & -K & \underline{2K} & -K & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & \dots & 0 & -K & \underline{2K} & -K & 0 \\ 0 & \dots & \dots & \dots & 0 & -K & \underline{2K} & 0 \\ 0 & \dots & 0 & 1 & -1 & 0 & \dots & 0 & 0 & \underline{0} \end{array} \right. ,$$

where  $K = k + n$ , the leftmost column indicates labeling of rows, and diagonal elements are underlined to guide the eye. The screenings that represent  $\mathfrak{n}_q \mathfrak{sl}(n|1)$  are given by

$$\begin{aligned} E_i &= \oint e^{(a_i, \varphi)}, \quad i = 1, \dots, n-m-1, \\ \Psi_+ &= \oint e^{(\psi_+, \varphi)}, \quad \Psi_- = \oint e^{(\psi_-, \varphi)}, \\ E_i &= \oint e^{(a_i, \varphi)}, \quad i = -1, \dots, -m+1. \end{aligned}$$

The  $n[m]$  realization of  $\mathcal{W}_n^{(2)}$ , denoted by  $\mathcal{W}_{n[m]}^{(2)}$ , is the centralizer of these operators in  $\mathcal{V}_\xi = \{\mathcal{P}(\partial\varphi) e^{m(\xi, \varphi)} \mid m \in \mathbb{Z}\}$ . As before,  $E_i$  are said to be bosonic and  $\Psi_\pm$  fermionic screenings.

The information contained in the Gram matrix above can be conveniently reexpressed as a “rigged” Dynkin diagram for  $\mathfrak{sl}(n|1)$ ,

$$(2.7) \quad \begin{array}{cccccccccccc} \bullet & \xrightarrow{-K} & \bullet & \xrightarrow{-K} & \dots & \xrightarrow{-K} & \bullet & \xrightarrow{-K} & \circ & \xrightarrow{K-1} & \circ & \xrightarrow{-K} & \bullet & \xrightarrow{-K} & \dots & \xrightarrow{-K} & \bullet \\ \text{\scriptsize } 2K & & \text{\scriptsize } 2K & & & & \text{\scriptsize } 2K & & \text{\scriptsize } 1 & & \text{\scriptsize } 1 & & \text{\scriptsize } 2K & & & & \text{\scriptsize } 2K \\ A_{n-m-1} & A_{n-m-2} & \dots & A_1 & Q_+ & Q_- & A_{-1} & \dots & A_{1-m} \end{array}$$

where  $K = k + n$ . The vertices of the diagram are assigned the dimension-1 currents  $A_{n-m-1}(z), \dots, A_1(z), Q_+(z), Q_-(z), A_{-1}(z), \dots, A_{-m+1}(z)$  as indicated. The labels  $2K$ ,  $-K$ , and  $1$  at the links mean that these currents have the OPEs

$$\begin{aligned} A_i(z) A_i(w) &= \frac{2K}{(z-w)^2}, & A_i(z) A_{i+1}(w) &= \frac{-K}{(z-w)^2}, \\ Q_\pm(z) Q_\pm(w) &= \frac{1}{(z-w)^2}, & Q_+(z) Q_-(w) &= \frac{K-1}{(z-w)^2}, \end{aligned}$$

$$A_1(z) Q_+(w) = \frac{-K}{(z-w)^2}, \quad A_{-1}(z) Q_-(w) = \frac{-K}{(z-w)^2}.$$

In addition, for the current  $Y = (\xi, \partial\varphi)$ , which is not associated with a vertex in the diagram, we have the nonzero OPEs (as indicated by the  $+$  and  $-$  signs),

$$Q_\pm(z) Y(w) = \frac{\pm 1}{(z-w)^2}.$$

**2.2. Centralizer of the screenings: the “step-back” strategy.** We now describe the strategy that leads to the construction of the  $\mathcal{W}_{n[m]}^{(2)}$  generators in the centralizer of the corresponding screenings. The contents of this subsection is not a proof, but because it motivates our construction, we hope that it can be useful in finding centralizers of some other quantum supergroups (a necessary condition is the existence of fermionic screening(s)). The crucial point is the conditions on the vector  $\xi$  (see (2.2), (2.6), and similar conditions read off from the matrix in **2.1.5**), chosen from quantum group considerations.

Quite generally, we recall that the action of screening operators  $S_1, \dots, S_N \in \mathcal{U}_q \mathfrak{n}$  on an operator  $X = \mathcal{P}' V(p', z)$  in a lattice vertex-operator algebra (where  $V(p', z)$  is a vertex with momentum  $p'$  and  $\mathcal{P}'$  is a differential polynomial) generically gives nonlocal expressions which represent elements of a module  $\mathcal{K}$  over the quantum group  $\mathcal{U}_q \mathfrak{g}$ . This module can be either a Verma module or (which is most often the case) some of its quotients. But whenever a singular vector, e.g.,  $v = S_1 \dots S_r X$  (where we write a monomial expression for simplicity) occurs in  $\mathcal{K}$ , the corresponding field is *local*—a descendant of the “shifted” vertex  $V(p, z)$ , where the momentum  $p$  differs from  $p'$  by the sum of the momenta of the relevant screenings,  $p = p' + a_1 + \dots + a_r$ . We are interested in the case where a singular vector in  $\mathcal{K}$  generates a *one-dimensional submodule*. Then the corresponding *local* field is necessarily *in the centralizer* of the quantum group. To construct a local field of the form  $\mathcal{P} V(p, z)$ , we must then start with an appropriate  $p' = p - a_1 - \dots - a_r$ .

That is, we seek a quantum group module  $\mathcal{K}$  with a singular vector  $v$  that generates a *one-dimensional submodule* in  $\mathcal{K}$  (the singular vector must therefore have weight 0). Such singular vectors (and hence the corresponding local fields) can indeed be found for the  $\mathfrak{sl}(n|1)$  root system (the existence of odd roots is a necessary condition). We now use this method to construct fields in the centralizer of  $\mathcal{U}_q \mathfrak{sl}(n|1)$ .

**2.2.1. Maximally asymmetric realization.** We first consider the maximally asymmetric realization, corresponding to the  $\mathfrak{sl}(n|1)$  simple root system described in **2.1.1**. We must find a Verma module quotient with a singular vector generating a one-dimensional submodule. For this, we take the  $\mathcal{U}_q \mathfrak{sl}(n|1)$ -module induced from the one-dimensional representation of the parabolic subalgebra generated by  $\mathcal{U}_q \mathfrak{gl}(n)$  and the fermionic simple root generator. The module is then isomorphic to  $\bigwedge^\bullet \mathbb{C}^n$  as a vector space. Its weight

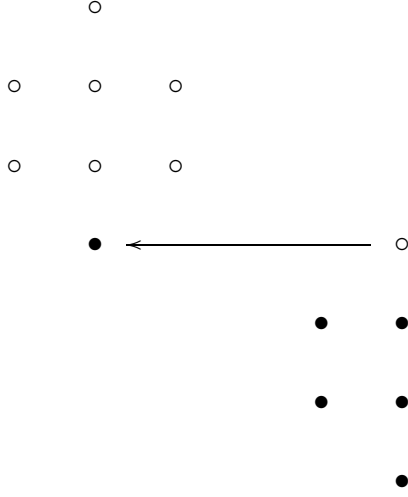


FIGURE 1. Mapping between two  $sl(3|1)$  modules whose highest-weight vectors are annihilated by the  $sl(3)$  subalgebra. Filled dots denote elements of *submodules*. Those in the right module ( $M_3(0)$ ) vanish under the mapping into the left module ( $M_3(-2)$ ).

diagram is shown in Fig. 1 for  $n = 3$  (the upper-left part, with the highest-weight vector at the top). For generic  $q$ , such a  $\mathcal{U}_q sl(n|1)$ -module is a deformation of an  $sl(n|1)$ -module and is isomorphic to the latter as a vector space, and we proceed with the corresponding  $sl(n|1)$ -module for simplicity. Such  $sl(n|1)$ -modules depend on a single parameter, the eigenvalue  $\alpha$  of the  $gl(n)$  Cartan generator that is not in  $sl(n)$ . We normalize this generator as

$$h_0 = \begin{pmatrix} \frac{1}{n} & & & \\ & \ddots & & \\ & & \frac{1}{n} & \\ & & & 1 \end{pmatrix}$$

and let  $|\alpha\rangle$  be the highest-weight vector of the above module with  $h_0|\alpha\rangle = \alpha|\alpha\rangle$ ; the module is denoted by  $M_n(\alpha)$ . *The vector at the bottom of the weight diagram of  $M_n(\alpha)$  is then singular if and only if  $\alpha = -(n-1)$ .*

In particular, the vector at the bottom of the weight diagram of  $M_n(1-n)$  is annihilated by  $h_0$ , and the highest-weight vector of the  $M_n(0)$  module is then mapped onto this singular vector, see Fig. 1. This picture is preserved under deformation to  $\mathcal{U}_q sl(n|1)$ . By the correspondence between modules over the quantum-group and the vertex operator algebra, the singular vector in (the deformation of)  $M_n(1-n)$  determines an intertwining operator between the corresponding  $\mathcal{W}_{n[0]}^{(2)}$  representations, realized in a sum of Fock spaces; this intertwining operator is morally the product of the  $n$  fermionic screenings  $\Psi$ ,  $\Psi', \dots, \Psi^{(n-1)}$  obtained by the action of the bosonic screenings on  $\Psi$ ,

$$\Psi \Psi' \dots \Psi^{(n-1)} : F_{\mathbf{p}'} \rightarrow F_{\mathbf{p}},$$

where  $F_{\mathbf{p}}$  is the module generated from  $e^{\mathbf{p} \cdot \varphi}$ . The image of the highest-weight vector in  $F_{\mathbf{p}'}$  is a descendant of the highest-weight vector in  $F_{\mathbf{p}}$ ,

$$(2.8) \quad (\Psi \Psi' \dots \Psi^{(n-1)})(e^{\mathbf{p}' \cdot \varphi}) = \mathcal{P}_n e^{\mathbf{p} \cdot \varphi},$$

where  $\mathcal{P}_n$  is a degree- $n$  differential polynomial in  $\partial\varphi = (\partial\varphi_1, \dots, \partial\varphi_n)$ . The difference  $\mathbf{p} - \mathbf{p}'$  must be equal to the sum of the momenta of  $\Psi$ ,  $\Psi'$ ,  $\dots$ ,  $\Psi^{(n-1)}$ , which are  $\psi$  for the fermionic screening in (2.3) and  $\psi + \mathbf{a}_1 + \dots + \mathbf{a}_i$  for each of the other fermionic screenings obtained by dressing  $\Psi$  with the bosonic screenings. Therefore,

$$(2.9) \quad \mathbf{p} = \mathbf{p}' + n\psi + (n-1)\mathbf{a}_1 + (n-2)\mathbf{a}_2 + \dots + \mathbf{a}_{n-1} = \mathbf{p}' - n\ell_n(k)\xi,$$

with  $\xi$  defined in (2.4).

Next, to ensure that the module is induced from the one-dimensional representation of the parabolic subalgebra, we require that

$$\mathbf{p} \cdot \mathbf{a}_i = 0, \quad i = 1, \dots, n-1$$

(with the scalar products among  $\mathbf{a}_i$  and  $\psi$  given in **2.1.1**, this is equivalent to  $\mathbf{p}' \cdot \mathbf{a}_i = 0$ ). Therefore,  $[E_i, e^{\mathbf{p} \cdot \varphi}] = 0$  and  $[E_i, e^{\mathbf{p}' \cdot \varphi}] = 0$ . Further, the vector in  $\mathcal{K}$  represented by the bottom dot in the upper-left module in Fig. 1 is singular, and the module is therefore (the quantum deformation of)  $M_n(1-n)$ , if

$$\mathbf{p} \cdot \psi = -1.$$

With these conditions satisfied, applying  $(\Psi \Psi' \dots \Psi^{(n-1)})$  to  $e^{\mathbf{p}' \cdot \varphi(z)}$  gives a level- $n$  descendant  $\mathcal{P}_n e^{\mathbf{p} \cdot \varphi(z)}$  of  $e^{\mathbf{p} \cdot \varphi(z)}$  that necessarily commutes with the screenings.<sup>4</sup> It follows that  $\mathbf{p} = -\xi$ , see (2.4). The currents

$$\begin{aligned} \overline{\mathcal{E}}_{n[0]}(z) &= e^{\xi \cdot \varphi(z)}, \\ \overline{\mathcal{F}}_{n[0]}(z) &= -\mathcal{P}_n e^{-\xi \cdot \varphi(z)} \end{aligned}$$

then generate  $\overline{\mathcal{W}}_{n[0]}^{(2)}$ . To recover  $\mathcal{W}_{n[0]}^{(2)}$ , it remains to introduce an additional free scalar field and embed the above  $n$ -dimensional vectors in  $\mathbb{C}^{n+1}$  as  $a_i = \{\mathbf{a}_i, 0\}$ ,  $\psi = \{\psi, 0\}$ , and  $\xi = \{\xi, \xi_{n+1}\}$  with *isotropic*  $\xi$ , as in **2.1.1**. We introduce the currents

$$(2.10) \quad A_i = (a_i, \partial\varphi), \quad Q = (\psi, \partial\varphi)$$

and the scalar field

$$\Xi = (\xi, \varphi)$$

---

<sup>4</sup>More generally, we expect that for generic  $k$ , the entire centralizer of  $\mathcal{U}_{qsl}(n|1)$  in the sector with momentum  $\mathbf{p}$  is given by  $(\Psi \Psi' \dots \Psi^{(n-1)})(\mathcal{P}' e^{\mathbf{p}' \cdot \varphi(z)})$ , where  $\mathcal{P}'$  is a differential polynomial such that  $[E_i, \mathcal{P}' e^{\mathbf{p}' \cdot \varphi}] = 0$ .

such that

$$(2.11) \quad Y = \partial \Xi = (\xi, \partial \varphi).$$

From the scalar product  $(\psi, \xi) = 1$ , we then have the OPEs

$$Q(z) e^{\pm \Xi(w)} = \frac{\pm 1}{z - w} e^{\pm \Xi(w)}.$$

In the “maximally asymmetric” realization, the  $\mathcal{W}_{n[0]}^{(2)}$ -currents have the form

$$\begin{aligned} \mathcal{E}_{n[0]}(z) &= e^{\Xi}(z), \\ \mathcal{F}_{n[0]}(z) &= -\mathcal{P}_n(A_{n-1}, \dots, A_1, Q) e^{-\Xi}(z), \end{aligned}$$

where  $\mathcal{P}_n(A_{n-1}, \dots, A_1, Q)$  is the polynomial in (2.8) expressed through  $A_{n-1}, \dots, A_1, Q$ .

**2.2.2. Other realizations.** In other realizations, both the  $\mathcal{E}$  and  $\mathcal{F}$  currents are given by a normal-ordered product of an exponential and a differential polynomial. We recall that in the well-known “symmetric” bosonization of  $\widehat{s\ell}(2)$  (see A.2), the  $\mathcal{E}$  and  $\mathcal{F}$  currents follow by the action of the corresponding fermionic screening and are therefore given by order-1 polynomials in front of the exponentials, see (A.3). In the general case, the simplest singular vectors given by the action of a single  $\mathcal{U}_q s\ell(2|1)$  generator are replaced with singular vectors in the appropriate  $\mathcal{U}_q s\ell(n|1)$  modules.

A “step-back” strategy similar to the one used in the maximally asymmetric case also involves  $2^n$ -dimensional  $\mathcal{U}_q s\ell(n|1)$ -modules; the classical  $s\ell(n|1)$ -analogue of every such module can be viewed as an  $M_n(1 - n)$  module turned on its side. This is illustrated in Fig. 2 for  $n = 4$ . If the module is viewed as generated from the top vector, it is  $M_4(-3)$ . In this case, the simple root generators correspond to the Dynkin diagram (a). But with the simple root generators corresponding to the Dynkin diagram (b), the  $2^4$ -dimensional  $s\ell(4|1)$  module is generated from the vector  $s$  east-south-east of the top one. The vector at the bottom “in the  $M_4(-3)$  coordinates” is then again singular and generates a one-dimensional submodule. Moreover, restricting to the  $s\ell(3|1)$  subalgebra (corresponding to the first three nodes in diagram (b)), we see that the  $s\ell(3|1)$ -module generated from  $s$  (shown with diamonds in Fig. 2) is  $M_3(-2)$ , as in the left-hand part of Fig. 1. The filled dot is also a singular vector in  $M_3(-2)$ , with a one-dimensional submodule (it is also clear how the  $s\ell(2|1)$ -module  $M_2(-1)$  fits this picture (bigger diamonds). Construction of the singular vector from the  $s$  state, actually *via the  $s\ell(3|1)$  generators*, then translates into a formula for the  $\mathcal{F}$  current in the centralizer of the screenings.

Moreover, the  $\mathcal{E}$  current in the centralizer of the screenings then follows by dressing the corresponding exponential with a single simple root generator; in Fig. 2, the corresponding module should be viewed as generated from the state  $s'$ , at “distance 1” from the singular vector with a 1-dimensional submodule.

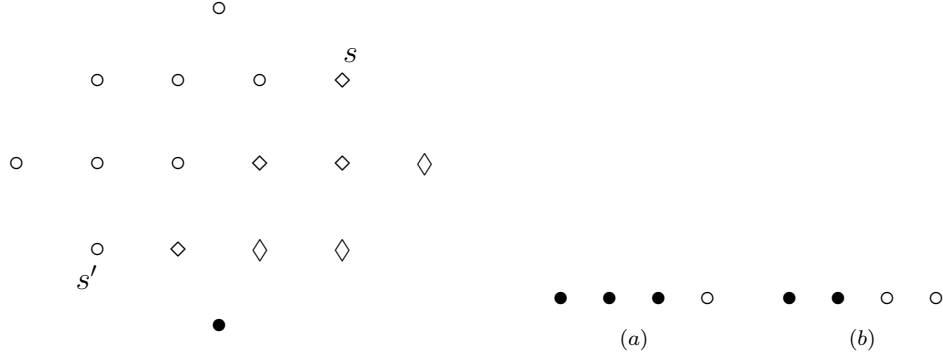


FIGURE 2. A “matrioshka” arrangement of special  $sl(n|1)$ -modules (*left*). Considered as generated from the top vector by the simple root generators corresponding to the Dynkin diagram (a), the  $2^4$  states represent the  $sl(4|1)$ -module  $M_4(-3)$ . The diamonds show the  $sl(3|1)$ -module  $M_3(-2)$ , as in the left part of Fig. 1. The filled dot is a singular vector in both  $M_4(-3)$  and  $M_3(-2)$ . The bigger diamonds (together with  $\bullet$ , which is again a singular vector) form the weight diagram of the  $sl(2|1)$ -module  $M_2(-1)$ . With the simple root generators corresponding to the Dynkin diagram (b), the  $2^4$ -dimensional  $sl(4|1)$  module is generated from the state  $s$ .

This hierarchy of the special  $sl(n|1)$ -modules under consideration (in fact, of their  $\mathcal{U}_q sl(n|1)$ -deformations) shows that singular vectors generating one-dimensional submodules can be constructed via the same step-back mechanism as in the maximally asymmetric case described above, but applied to a subset of simple root generators. The singular vector is in a sense “the same” for the different root systems in the  $sl(n|1)$  algebras with different  $n$ , only constructed from differently chosen highest-weight vectors by the appropriate set of simple root generators. Accordingly, the  $\mathcal{E}$  and  $\mathcal{F}$  currents in the centralizer of the screenings are then obtained by dressing the corresponding exponential with *complementary* sets of the simple root generators,  $E_i, i = 1, \dots, n - m - 1$ , and  $\Psi_+$  for  $\mathcal{F}_{n[m]}$ , and  $E_i, i = -1, \dots, -m + 1$  and  $\Psi_-$  for  $\mathcal{E}_{n[m]}$ , and therefore,

$$\begin{aligned}\mathcal{E}_{n[m]}(z) &= \mathcal{P}_m^\dagger(A_{-1}, \dots, A_{-m+1}, Q_-) e^\Xi(z), \\ \mathcal{F}_{n[m]}(z) &= (-)^{m+1} \mathcal{P}_{n-m}(A_{n-m-1}, \dots, A_1, Q_+) e^{-\Xi(z)}\end{aligned}$$

with differential polynomials  $\mathcal{P}_m^\dagger$  and  $\mathcal{P}_{n-m}$  of the respective order  $m$  and  $n - m$  (normal ordering in the right-hand sides is understood).

**2.3. Centralizer of  $\mathcal{U}_q sl(n|1)$ : recursion relations.** Parallel to the “matrioshka” arrangement of the  $sl(n|1)$  modules described in the previous subsection, there exist recursion relations between the differential polynomials entering the different realizations



of the  $\mathcal{W}_n^{(2)}$  algebras with different  $n$ . In considering these, we must be very precise about notation.

**2.3.1. Notational chores.** We write  $\mathcal{X}_{n[m]}^{(k)}$  for generators  $\mathcal{X}_{n[m]} = (\mathcal{E}_{n[m]}, \mathcal{H}_{n[m]}, \mathcal{F}_{n[m]})$  of  $\mathcal{W}_{n[m]}^{(2)}(k)$  whenever we need to indicate the level  $k$  of the algebra. The differential polynomials  $\mathcal{P}_{n[m]}^\dagger$  and  $\mathcal{P}_{n[m]}$  in  $A_i$  and  $Q_\pm$  entering these generators depend on  $k$  explicitly, which we indicate by writing them as  $\mathcal{P}_{n[m]}^{\dagger(k)}$  and  $\mathcal{P}_{n[m]}^{(k)}$ . Moreover, the free fields introduced in 2.1.5 also bear an implicit dependence on  $k$  and  $n$ , in fact on  $k+n$  involved in their OPEs (where we recall that  $K = k+n$ ). When we need to be very precise, we write these free fields as  $A_i^{[k+n]}$  and  $Q_\pm^{[k+n]}$ , and similarly use the notation  $E_i^{[k+n]}$  and  $\Psi_\pm^{[k+n]}$  for the screenings. In the  $n[m]$  realization of  $\mathcal{W}_n^{(2)}(k)$ , we can thus write the generators, most generally, as (see (2.10)–(2.11) for the definition of the currents)

$$(2.12) \quad \begin{aligned} \mathcal{E}_{n[m]}^{(k)}(z) &= \mathcal{P}_{n[m]}^{\dagger(k)}(A_\bullet^{[k+n]}, Q_+^{[k+n]}, Q_-^{[k+n]}) e^{\Xi(z)}, \\ \mathcal{F}_{n[m]}^{(k)}(z) &= (-1)^{m+1} \mathcal{P}_{n[m]}^{(k)}(A_\bullet^{[k+n]}, Q_+^{[k+n]}, Q_-^{[k+n]}) e^{-\Xi(z)}, \end{aligned}$$

where  $\mathcal{P}_{n[m]}^{\dagger(k)}$  and  $\mathcal{P}_{n[m]}^{(k)}$  are differential polynomials of the respective degrees  $m$  and  $n-m$  and  $A_\bullet$  stands for the appropriate collection of the  $A_i$  currents (see 2.1.5). The conventional sign factor is chosen for future convenience. By definition,  $\mathcal{P}_{n[0]}^\dagger = \mathcal{P}_{n[n]} = 1$ . For  $m = 0$  (the maximally asymmetric realization),  $Q_-$  does not enter and  $Q_+$  is identified with  $Q$  in 2.1.1.

**2.3.2. Example.** From the well-known three-boson realizations of  $\widehat{s\ell}(2)$ , see A.2, we have that the lowest  $\mathcal{P}$  polynomials are given by

$$(2.13) \quad \begin{aligned} \mathcal{P}_{2[1]}^{\dagger(k)}(Q) &= \mathcal{P}_{2[1]}^{(k)}(Q) = Q, \\ \mathcal{P}_{2[0]}^{(k)}(A, Q) &= AQ + QQ + (k+1)\partial Q. \end{aligned}$$

**2.3.3. Field identifications.** It follows that the free fields entering a realization of the  $\mathcal{W}_{n-1}^{(2)}(k+1)$  algebra can be *identified* with a subset of the fields involved in a realization of  $\mathcal{W}_n^{(2)}(k)$ . We can therefore consider the “universal” sets of the currents

$$\dots, A_2^{[\kappa]}, A_1^{[\kappa]}, Q_+^{[\kappa]}, Q_-^{[\kappa]}, A_{-1}^{[\kappa]}, A_{-2}^{[\kappa]}, \dots$$

and the screenings

$$\dots, E_2^{[\kappa]}, E_1^{[\kappa]}, \Psi_+^{[\kappa]}, \Psi_-^{[\kappa]}, E_{-1}^{[\kappa]}, E_{-2}^{[\kappa]}, \dots,$$

and use their appropriate finite subsets to define the  $\mathcal{W}_{n[m]}^{(2)}$ . Any such subset is a length- $n$  segment including  $\Psi_+$  or  $\Psi_-$  (or both).<sup>5</sup>

<sup>5</sup>The case where only  $\Psi_+$ , but not  $\Psi_-$  is included is the maximally asymmetric realization with  $m = 0$ . The case where only  $\Psi_-$  is included, but  $\Psi_+$  is not, is the “equally asymmetric,” opposite realization with

Next, the scalar products of  $\xi$  with the other vectors (see **2.1.5**), and hence the OPEs of  $Y = (\xi, \partial\varphi)$  with the other currents are independent of  $k$  or  $n$ ; this allows us to *identify the  $\Xi$  and  $Y$  fields in all the realizations of all  $\mathcal{W}_n^{(2)}$* .

We now find the dimension-1 current in each  $\mathcal{W}_{n[m]}^{(2)}$ .

**2.3.4. Lemma.** *For  $0 \leq m \leq n$ ,  $n \geq 2$ , the diagonal current of  $\mathcal{W}_{n[m]}^{(2)}$  is given by*

$$\begin{aligned} \mathcal{H}_{n[m]}^{(k)}(z) = \ell_n(k) Y(z) + \sum_{i=1}^{n-m-1} \frac{n-i-m}{n} A_i^{[k+n]}(z) + \frac{n-m}{n} Q_+^{[k+n]}(z) \\ - \sum_{i=-m+1}^{-1} \frac{m+i}{n} A_i^{[k+n]}(z) - \frac{m}{n} Q_-^{[k+n]}(z). \end{aligned}$$

*Proof.* This is shown by directly finding the centralizer of the relevant screening operators in the space of dimension-1 operators with zero “momentum” — i.e., among descendants of the identity operator, and hence necessarily linear combinations of the currents; the coefficients are then determined by a straightforward calculation, uniquely up to normalization, and the overall normalization is fixed by (1.2).  $\square$

Proceeding similarly, we establish the existence of a Virasoro algebra in the centralizer of the screenings:

**2.3.5. Lemma.** *For generic  $k$ , the centralizer of the screenings  $E_i$ ,  $i = 1, \dots, n-1$ , and  $\Psi$  (see (2.3)) in the space of dimension-2 operators that are descendants of the identity is three-dimensional. In addition to  $\mathcal{H}\mathcal{H}(z)$  and  $\partial\mathcal{H}(z)$ , it is generated by the energy-momentum tensor with central charge (1.1), explicitly given by*

$$\begin{aligned} \mathcal{T}_n^{(k)}(z) = \frac{1}{2} \sum_{i,j \in \{n-1, n-2, \dots, 1, +, *\}} (\Gamma_n(k)^{-1})_{i,j} A_i^{[k+n]} A_j^{[k+n]}(z) \\ + \sum_{i=1}^{n-1} (n-i) \frac{(i-1)(k+n-1)-1}{2(k+n)} \partial A_i^{[k+n]}(z) - \frac{n}{2} \partial A_+^{[k+n]}(z), \end{aligned}$$

where we write  $A_+ \equiv Q_+ \equiv Q$  and  $A_* \equiv Y$  for notational uniformity and where  $\Gamma_n(k)$  is the “dressed” Cartan matrix (2.5).

Lemma **2.3.4** implies a relation between the  $\mathcal{H}$  currents in  $\mathcal{W}_{n[0]}^{(2)}(k)$  and  $\mathcal{W}_{(n-1)[0]}^{(2)}(k+1)$ :

$$(2.14) \quad n\mathcal{H}_{n[0]}^{(k)} - (n-1)\mathcal{H}_{(n-1)[0]}^{(k+1)} = (k+n-1)Y + \sum_{i=1}^{n-1} A_i^{[k+n]} + Q^{[k+n]}.$$

---

$m = n$ . We do not consider it specially because it can be obtained from the  $m = 0$  realization by the automorphism exchanging  $\mathcal{E}$  and  $\mathcal{F}$ .

Somewhat less obviously, we have similar recursion relations for the  $\mathcal{E}$  and  $\mathcal{F}$  currents generating  $\mathcal{W}_{n[0]}^{(2)}$ .

**2.3.6. Lemma.** *Let  $\mathcal{E}_{(n-1)[0]}^{(k+1)} = e^\Xi$ ,  $\mathcal{H}_{(n-1)[0]}^{(k+1)}$ , and  $\mathcal{F}_{(n-1)[0]}^{(k+1)} = -\mathcal{P}_{n-1}^{(k+1)} e^{-\Xi}$  be the generators of  $\mathcal{W}_{(n-1)[0]}^{(2)}(k+1)$ ,  $n \geq 2$ . Then*

$$\mathcal{F}_{n[0]}^{(k)} = ((k+n-1)\partial + n\mathcal{H}_{n[0]}^{(k)} - (n-1)\mathcal{H}_{n-1}^{(k+1)})\mathcal{F}_{(n-1)[0]}^{(k+1)}, \quad n \geq 2,$$

*is in the centralizer of the screenings  $(E_i)_{i=1,\dots,n-1}$  and  $\Psi$ , i.e., in  $\mathcal{W}_{n[0]}^{(2)}(k)$ . In other words,  $\mathcal{F}_{n[0]}^{(k)} = -\mathcal{P}_n^{(k)} e^{-\Xi}$  can be constructed recursively, via*

$$\mathcal{P}_n^{(k)} = \left( (k+n-1)\partial + Q^{[k+n]} + \sum_{i=1}^{n-1} A_i^{[k+n]} \right) \mathcal{P}_{n-1}^{(k+1)}$$

*with the initial condition  $\mathcal{P}_1^{(k)}(Q) = Q$ .*

In accordance with a remark in **2.3.1**, this recursion relies on identification of the fields  $A_{n-2}, \dots, A_1, Q$  involved in the construction of  $\mathcal{W}_{(n-1)[0]}^{(2)}(k+1)$  with the corresponding fields among the  $A_{n-1}, A_{n-2}, \dots, A_1, Q$  involved in the construction of  $\mathcal{W}_{n[0]}^{(2)}(k)$ . As a corollary, we note that the differential polynomial  $\mathcal{P}_n$  depends only on  $A_{n-1}, \dots, A_1, Q$ .

*Proof of 2.3.6.* The second formula in **2.3.6** is equivalent to the first one in view of (2.14). We now show that  $\mathcal{F}_{n[0]}^{(k)}$  constructed via the recursion is in the centralizer of the screenings (2.3). We assume that this is so for the *preceding* generators, in particular,

$$\begin{aligned} [E_i^{[k+n]}, \mathcal{F}_{(n-1)[0]}^{(k+1)}] &= 0, \quad i = 1, \dots, n-2, \\ [\Psi^{[k+n]}, \mathcal{F}_{(n-1)[0]}^{(k+1)}] &= 0, \end{aligned}$$

or, in terms of OPEs for  $i = 1, \dots, n-2$ ,

$$(2.15) \quad e^{(a_i, \varphi(z))} \mathcal{P}_{n-1}^{(k+1)}(A_{n-2}^{[k+n]}, \dots, A_1^{[k+n]}, Q^{[k+n]})(w) = \frac{0}{z-w} + \dots,$$

where the dots denote possible other poles (only the vanishing of the indicated pole is essential), and

$$(2.16) \quad e^{(\psi, \varphi(z))} \mathcal{P}_{n-1}^{(k+1)}(A_{n-2}^{[k+n]}, \dots, A_1^{[k+n]}, Q^{[k+n]})(w) = 0 \cdot (z-w)^0 + \dots$$

(that is, the normal-ordered product vanishes).<sup>6</sup> To be precise, we should have written  $\varphi^{[k+n]}$  in the left-hand sides.

---

<sup>6</sup>For  $n-1=1$ , with  $\mathcal{P}_1(Q)=Q$ , the vanishing in (2.16) occurs as follows: the OPE

$$e^{(\psi, \varphi(z))} Q(w) = -\frac{e^{(\psi, \varphi(z))}}{z-w} + :e^{(\psi, \varphi)} Q:(w),$$

gives zero in the normal-ordered term after the expansion  $e^{(\psi, \varphi(z))} = e^{(\psi, \varphi(w))} + (z-w)Q(w)$ .

We must show that relations (2.15) and (2.16) imply the OPEs

$$(2.17) \quad e^{(a_i, \varphi(z))} \mathcal{P}_n^{(k)}(A_{n-1}^{[k+n]}, \dots, A_1^{[k+n]}, Q^{[k+n]})(w) = \frac{0}{z-w} + \dots, \quad i = 1, \dots, n-1,$$

and

$$(2.18) \quad e^{(\psi, \varphi(z))} \mathcal{P}_n^{(k)}(A_{n-1}^{[k+n]}, \dots, A_1^{[k+n]}, Q^{[k+n]})(w) = 0 \cdot (z-w)^0 + \dots$$

But we readily establish the OPE

$$A_i(z) \left( Q(w) + \sum_{j=1}^{n-1} A_j(w) \right) = 0, \quad i = 1, \dots, n-2,$$

and therefore (2.15) and the recursion for  $\mathcal{P}$  imply (2.17) for  $i = 1, \dots, n-2$ . For  $i = n-1$ , we no longer have a similar vanishing OPE, but *the only* nonzero OPEs involving  $A_{n-1}$  are with itself and with  $A_{n-2}$ , see (2.7). The dependence of  $\mathcal{P}_n^{(k)}$  on  $A_{n-1}$  is only through the explicit occurrences of  $A_{n-1}$  in the recursion relation, and the dependence on  $A_{n-2}$  is only through the explicit occurrences of  $A_{n-2}$  in the *preceding* recursion relation. That is,

$$\begin{aligned} \mathcal{P}_n^{(k)} = & (k+n-1) \partial \left( (k+n-1) \partial \mathcal{P}_{n-2}^{(k+2)} + (A_{n-2} + \sum_{j=1}^{n-3} A_j + Q) \mathcal{P}_{n-2}^{(k+2)} \right) \\ & + (A_{n-1} + A_{n-2} + \sum_{j=1}^{n-3} A_j + Q) \left( (k+n-1) \partial \mathcal{P}_{n-2}^{(k+2)} \right. \\ & \left. + (A_{n-2} + \sum_{j=1}^{n-3} A_j + Q) \mathcal{P}_{n-2}^{(k+2)} \right), \end{aligned}$$

where  $\mathcal{P}_{n-2}^{(k+2)}$  depends only on the fields that have zero OPEs with  $A_{n-1}$ . This shows that the first-order pole in the OPE  $e^{(a_{n-1}, \varphi(z))} \mathcal{P}_n^{(k)}(w)$  vanishes; the derivation is elementary. This proves all the equations (2.17).

It remains to verify (2.18), i.e.,  $:e^{(\psi, \varphi)} \mathcal{P}_n^{(k)}: = 0$ . For this, we note that  $Q$  (and hence  $e^{(\psi, \varphi)}$ ) has nonvanishing OPEs only with  $Q$  and  $A_1$ ; in particular, it follows from **2.1.5** that

$$e^{(\psi, \varphi(z))} \left( Q(w) + \sum_{j=1}^{n-1} A_j(w) \right) = (k+n-1) \frac{e^{(\psi, \varphi(w))}}{z-w}.$$

The recursion now readily implies that  $:e^{(\psi, \varphi)} \mathcal{P}_n^{(k)}: = (k+n-1) \partial :e^{(\psi, \varphi)} \mathcal{P}_{n-1}^{(k+1)}: = 0$  by assumption.  $\square$

**2.3.7. Example.** For  $n = 3$ , it follows from Lemma **2.3.6** and (2.13) that

$$\begin{aligned} \mathcal{P}_3^{(k)}(A_2, A_1, Q) = & A_1 A_1 Q + A_1 A_2 Q + 2A_1 Q Q + A_2 Q Q + Q Q Q \\ & + (k+2)(2A_1 \partial Q + A_2 \partial Q + \partial A_1 Q + 3\partial Q Q) + (k+2)^2 \partial^2 Q. \end{aligned}$$

The  $n = 4$  example is given in Appendix A.4.

Generalizing **2.3.6** from  $\mathcal{W}_{n[0]}^{(2)}$  to  $\mathcal{W}_{n[m]}^{(2)}$ , we have

**2.3.8. Lemma.** *Let the currents  $\mathcal{E}_{(n-1)[m]}^{(k+1)} = \mathcal{P}_{(n-1)[m]}^{\dagger(k+1)} e^{\Xi}$ ,  $\mathcal{H}_{(n-1)[m]}^{(k+1)}$ , and  $\mathcal{F}_{(n-1)[m]}^{(k+1)} = (-1)^{m+1} \mathcal{P}_{(n-1)[m]}^{(k+1)} e^{-\Xi}$  be the generators of  $\mathcal{W}_{(n-1)[m]}^{(2)}(k+1)$ ,  $n \geq m+2$ . Then  $\mathcal{F}_{n[m]}^{(k)} = (-1)^{m+1} \mathcal{P}_{n[m]}^{(k)} e^{-\Xi}$  with*

$$\mathcal{P}_{n[m]}^{(k)} = \left( (k+n-1)\partial + Q_+ + \sum_{i=1}^{n-m-1} A_i \right) \mathcal{P}_{(n-1)[m]}^{(k+1)}, \quad n \geq m+2,$$

and with the initial condition  $\mathcal{P}_{(m+1)[m]}^{(k)}(Q) = Q$  is in the centralizer of the screenings  $(E_i)_{i=1, \dots, n-m-1}$ ,  $\Psi_+$ ,  $\Psi_-$ ,  $(E_i)_{i=-1, \dots, -m+1}$ , i.e., in  $\mathcal{W}_{n[m]}^{(2)}(k)$ .

We note that this implies that  $\mathcal{P}_{n[m]}^{(k)}$  depends only on  $A_{n-m-1}, \dots, A_1, Q_+$ ; anticipating a similar statement for  $\mathcal{P}^\dagger$  (that the differential polynomial  $\mathcal{P}_{n[m]}^\dagger$  depends only on  $A_{-1}, \dots, A_{-m+1}, Q_-$  for  $m \geq 2$  (and on  $Q_-$  for  $m = 1$ )), we can express (2.12) much more precisely, as

$$(2.19) \quad \begin{aligned} \mathcal{E}_{n[m]}^{(k)}(z) &= \mathcal{P}_{n[m]}^{\dagger(k)}(A_{-1}^{[k+n]}, \dots, A_{-m+1}^{[k+n]}, Q_-^{[k+n]}) e^{\Xi(z)}, \\ \mathcal{F}_{n[m]}^{(k)}(z) &= (-1)^{m+1} \mathcal{P}_{n[m]}^{(k)}(A_{n-m-1}^{[k+n]}, \dots, A_1^{[k+n]}, Q_+^{[k+n]}) e^{-\Xi(z)}. \end{aligned}$$

*Proof of 2.3.8.* The only new element compared to the proof of **2.3.6** consists in verifying the vanishing of the *second-order* pole in the operator product involving  $\Psi_-$ ,

$$e^{(\psi_-, \varphi(z))} \mathcal{P}_{n[m]}^{(k)}(A_{n-m-1}, \dots, A_1, Q_+)(w) = \frac{0}{(z-w)^2} + \dots$$

assuming that this vanishing occurs for  $\mathcal{P}_{(n-1)[m]}^{(k+1)}$ . We use the notation  $[A, B]_n$  for the coefficient at the  $n$ th-order pole in the OPE  $A(z)B(w)$  and recall the standard relations (see, e.g., [9])

$$\begin{aligned} [V, \partial \mathcal{P}]_2 &= [V, \mathcal{P}]_1 + \partial[V, \mathcal{P}]_2, \\ [V, [\mathcal{A}, \mathcal{P}]_0]_2 &= [\mathcal{A}, [V, \mathcal{P}]_2]_0 + \sum_{\ell > 0} [[V, \mathcal{A}]_\ell, \mathcal{P}]_{2-\ell}, \end{aligned}$$

where we take  $V = e^{(\psi_-, \varphi)}$ ,  $\mathcal{A} = Q_+ + \sum_{i=1}^{n-m-1} A_i$ , and  $\mathcal{P} = \mathcal{P}_{(n-1)[m]}^{(k+1)}$ . The assumption is therefore that  $[V, \mathcal{P}]_2 = 0$ . It now follows from **2.1.5** that  $V(z)\mathcal{A}(w) = \frac{[V, \mathcal{A}]_1}{z-w}$  with  $[V, \mathcal{A}]_1 = -(k+n-1)V$ , and therefore

$$\begin{aligned} [e^{(\psi_-, \varphi)}, \mathcal{P}_{n[m]}^{(k)}]_2 &= [V, (k+n-1)\partial \mathcal{P} + [\mathcal{A}, \mathcal{P}]_0]_2 = \\ &= (k+n-1)[V, \mathcal{P}]_1 + [[V, \mathcal{A}]_1, \mathcal{P}]_1 = 0, \end{aligned}$$

which is equivalent to the statement that  $\mathcal{P}_{n[m]}^{(k)} e^{-\Xi}$ , with  $\mathcal{P}_{n[m]}^{(k)}$  given by the recursion formula, is in the centralizer of  $\Psi_-$ .  $\square$

As an immediate consequence of **2.3.6** and **2.3.8**, we have

**2.3.9. Lemma.** *For  $1 \leq m \leq n-1$ ,*

$$\mathcal{P}_{n[m]}^{(k)}(A_{n-m-1}, \dots, A_1, Q_+) = \mathcal{P}_{(n-1)[m-1]}^{(k+1)}(A_{n-m-1}, \dots, A_1, Q_+),$$

and therefore

$$\mathcal{P}_{n[m]}^{(k)} = \mathcal{P}_{n-m}^{(k+m)}.$$

Finally, the  $\mathcal{P}^\dagger$  polynomials are also expressed through the  $\mathcal{P}_m$ . This follows by reading the Dynkin diagram describing the  $n[m]$  realization from right to left, which corresponds to the algebra automorphism interchanging  $\mathcal{E}$  and  $\mathcal{F}$  (and also replacing  $m$  with  $n-m$ ).

**2.3.10. Lemma.** *For  $m = 1, \dots, n-1$ ,*

$$\mathcal{P}_{n[m]}^{\dagger(k)} = \mathcal{P}_m^{(k+n-m)}.$$

This shows, in particular, that the  $\mathcal{P}_{n[m]}^{\dagger(k)}$  depend on the currents as indicated in (2.19).

**2.4.  $\mathcal{W}_{n[m]}^{(2)}$  OPEs.** We thus see that all realizations of all the  $\mathcal{W}_\bullet^{(2)}$  algebras are determined by a series of degree- $n$  differential polynomials  $\mathcal{P}_n$  in  $n$  variables,  $n \geq 1$ , given by the *normal-ordered* expressions

$$\begin{aligned} \mathcal{P}_n^{(k)}(A_{n-1}, \dots, A_1, Q)(z) &= \\ &= \left( (k+n-1)\partial + Q(z) + \sum_{i=1}^{n-1} A_i \right) \circ \left( (k+n-1)\partial + Q(z) + \sum_{i=1}^{n-2} A_i(z) \right) \circ \\ &\quad \circ \dots \circ \left( (k+n-1)\partial + Q(z) + A_1(z) \right) Q(z). \end{aligned}$$

All the  $\partial \equiv \frac{\partial}{\partial z}$  operators are applied to the currents on the right. Linearly combining the currents as

$$\begin{aligned} R_0^+ &= Q_+, \\ R_i^+ &= Q_+ + A_1 + \dots + A_i, \quad i \geq 1, \\ R_0^- &= Q_-, \\ R_i^- &= Q_- + A_{-1} + \dots + A_{-i}, \quad i \geq 1, \end{aligned}$$

we readily see that their OPEs are indeed those in (1.4)–(1.5), and we obtain Theorem **1.1**.

The other currents in the algebra are to be found from the OPE  $\mathcal{E}_{n[m]}^{(k)}(z) \mathcal{F}_{n[m]}^{(k)}(w)$ . Calculating it, we first obtain

$$\begin{aligned} \mathcal{E}_{n[m]}^{(k)}(z) \mathcal{F}_{n[m]}^{(k)}(w) &= (-1)^{m+1} e^{\Xi(z) - \Xi(w)} \times \\ &\left[ ((k+n-1)\partial_z + R_{m-1}^-(z) + \frac{1}{z-w}) \dots \right. \\ &\quad \left. ((k+n-1)\partial_z + R_1^-(z) + \frac{1}{z-w})(R_0^-(z) + \frac{1}{z-w}) \right] \\ &\left[ ((k+n-1)\partial_w + R_{n-m-1}^+(w) - \frac{1}{z-w}) \dots \right. \\ &\quad \left. ((k+n-1)\partial_w + R_1^+(w) - \frac{1}{z-w})(R_0^+(w) - \frac{1}{z-w}) \right], \end{aligned}$$

where the OPEs with the exponentials have been taken into account and *it remains to evaluate the OPEs  $R_i^-(z)R_j^+(w)$  between the currents in the two (“ $z$ ” and “ $w$ ”) brackets.* The action of the  $\partial$  operators is delimited by the square brackets. Because the OPEs  $R_i^-(z)R_j^+(w)$  are independent of  $i$  and  $j$  (see (1.4)), we obtain

$$\begin{aligned} (2.20) \quad \mathcal{E}_{n[m]}^{(k)}(z) \mathcal{F}_{n[m]}^{(k)}(w) &= (-1)^{m+1} e^{\Xi(z) - \Xi(w)} \\ &\times \left[ ((k+n-1)\partial_z + R_{m-1}^-(z) + \frac{k+n-1}{(z-w)^2} \nabla_+ + \frac{1}{z-w}) \dots \right. \\ &\quad \left. ((k+n-1)\partial_z + R_1^-(z) + \frac{k+n-1}{(z-w)^2} \nabla_+ + \frac{1}{z-w}) \right. \\ &\quad \left. (R_0^-(z) + \frac{k+n-1}{(z-w)^2} \nabla_+ + \frac{1}{z-w}) \right] \\ &\left[ ((k+n-1)\partial_w + R_{n-m-1}^+(w) - \frac{1}{z-w}) \dots \right. \\ &\quad \left. ((k+n-1)\partial_w + R_1^+(w) - \frac{1}{z-w})(R_0^+(w) - \frac{1}{z-w}) \right] \Big|_{\nabla_+(1)=0}, \end{aligned}$$

where the right-hand side is normal-ordered and  $\nabla_+$  is the derivation of the ring of differential polynomials such that

$$\nabla_+(R_i^+) = 1.$$

After all  $\nabla_+$ ’s are evaluated in accordance with this rule, we must set  $\nabla_+ = 0$ , which is indicated by the prescription  $\nabla_+(1) = 0$ . As noted above, the action of  $\partial_z = \frac{\partial}{\partial z}$  is delimited by the first of the two groups of factors in brackets (the “ $z$ ”-factors). But each  $\partial_z$  is involved only in the combination  $\partial_z + \frac{1}{(z-w)^2} \nabla_+$ , where  $\nabla_+$  is then applied to  $R_i^+(w) - \frac{1}{z-w}$ . Because

$$\frac{1}{(z-w)^2} \nabla_+(R_i^+(w)) = \partial_z \left( -\frac{1}{z-w} \right),$$

it follows that the right-hand side of the OPE  $\mathcal{E}_{n[m]}^{(k)}(z) \mathcal{F}_{n[m]}^{(k)}(w)$  can be rewritten with all the  $\nabla_+$  dropped and with each  $\partial_z$  acting on *all* factors, either “ $z$ ” or “ $w$ ”, to the right of a

given one. Thus,

$$\begin{aligned}
(2.21) \quad \mathcal{E}_{n[m]}^{(k)}(z) \mathcal{F}_{n[m]}^{(k)}(w) &= (-1)^{m+1} e^{\Xi(z) - \Xi(w)} \\
&\times \left( (k+n-1) \partial_z + R_{m-1}^-(z) + \frac{1}{z-w} \right) \dots \\
&\left( (k+n-1) \partial_z + R_1^-(z) + \frac{1}{z-w} \right) \left( (k+n-1) \partial_z + R_0^-(z) + \frac{1}{z-w} \right) \\
&\left( (k+n-1) \partial_w + R_{n-m-1}^+(w) - \frac{1}{z-w} \right) \dots \\
&\left( (k+n-1) \partial_w + R_1^+(w) - \frac{1}{z-w} \right) \left( R_0^+(w) - \frac{1}{z-w} \right).
\end{aligned}$$

In this normal-ordered expression, we expand  $e^{\Xi(z) - \Xi(w)}$  up to the terms  $\mathcal{O}((z-w)^{n-1})$  and then rewrite it as

$$\mathcal{E}_{n[m]}^{(k)}(z) \mathcal{F}_{n[m]}^{(k)}(w) = \sum_{j=1}^n \frac{\mathcal{U}_{n[m],n-j}^{(k)}(w)}{(z-w)^j}.$$

This gives the central term  $\mathcal{U}_{n[m],0}^{(k)} = \lambda_{n-1}(n, k)$ , the dimension-1 current  $\mathcal{U}_{n[m],1}^{(k)}(w) = n\lambda_{n-2}(n, k)\mathcal{H}_{n[m]}^{(k)}(w)$ , the energy-momentum tensor related to  $\mathcal{U}_{n[m],2}^{(k)}(w)$  as in (A.1), and the other currents  $\mathcal{U}_{n[m],i}^{(k)}(w)$ ,  $3 \leq i \leq n-1$ .

Evaluating the integrals

$$\oint dz f(z) (\mathcal{E}_{n[m]}^{(k)}(z) \mathcal{F}_{n[m]}^{(k)}(w)) = \sum_{j=1}^n \frac{1}{(j-1)!} \mathcal{U}_{n[m],n-j}^{(k)}(w) \partial_w^{j-1} f(w)$$

with suitable test functions, we arrange the currents  $\mathcal{U}_{n[m],i}^{(k)}(w)$  into the order- $(n-1)$  differential operator

$$U_{n[m]}^{(k)} = \sum_{j=1}^n \frac{1}{(j-1)!} \mathcal{U}_{n[m],n-j}^{(k)} \partial^{j-1},$$

similarly to standard cases of the quantum Drinfeld–Sokolov reduction. Because of the presence of both derivatives and Cauchy kernels in the right-hand side of (2.21), the corresponding analogue of the Miura mapping is more complicated than in the classic cases. We only note that

$$U_{n[m]}^{(k)} = e^{-\Xi} V_{n[m]}^{(k)}(e^{\Xi} \cdot),$$



where  $\mathbf{V}_{n[m]}^{(k)}$  is the order- $(n-1)$  differential operator given by

$$\begin{aligned} (\mathbf{V}_{n[m]}^{(k)} f)(w) &= \oint dz f(z) \left( (k+n-1) \overleftarrow{\partial}_z - R_{m-1}^-(z) - \frac{1}{z-w} \right) \dots \\ &\quad \left( (k+n-1) \overleftarrow{\partial}_z - R_1^-(z) - \frac{1}{z-w} \right) \left( (k+n-1) \overleftarrow{\partial}_z - R_0^-(z) - \frac{1}{z-w} \right) \\ &\quad \times \left( (k+n-1) \partial_w + R_{n-m-1}^+(w) - \frac{1}{z-w} \right) \dots \\ &\quad \left( (k+n-1) \partial_w + R_1^+(w) - \frac{1}{z-w} \right) \left( R_0^+(w) - \frac{1}{z-w} \right). \end{aligned}$$

Derivatives acting on  $f$  are written on the right to simplify comparison with (2.21).

**2.4.1. Example.** Taking  $n = 2$  and  $m = 1$  brings us back to the symmetric realization of  $\widehat{s\ell}(2)$ . Equation (2.21) then becomes

$$\begin{aligned} \mathcal{E}_{2[1]}^{(k)}(z) \mathcal{F}_{2[1]}^{(k)}(w) &= e^{\Xi(z) - \Xi(w)} \left( (k+1) \partial_z + Q_-(z) + \frac{1}{z-w} \right) \left( Q_+(w) - \frac{1}{z-w} \right) \\ &= (1 + (z-w)Y) \left( \frac{Q_+ - Q_-}{z-w} + \frac{k}{(z-w)^2} \right) \\ &= \frac{k}{(z-w)^2} + \frac{Q_+ - Q_- + kY}{z-w}, \end{aligned}$$

with  $Q_+ - Q_- + kY = 2\mathcal{H}_{2[1]}^{(k)}$ .

Similarly, for the BP algebra ( $n = 3$ ) in the maximally asymmetric realiation ( $m = 0$ ), we evaluate (2.21) as

$$\begin{aligned} \mathcal{E}_{3[0]}(z) \mathcal{F}_{3[0]}(w) &= -e^{\Xi(z) - \Xi(w)} \left( (k+2) \partial_w + R_2^+(w) - \frac{1}{z-w} \right) \\ &\quad \left( (k+2) \partial_w + R_1^+(w) - \frac{1}{z-w} \right) \left( R_0^+(w) - \frac{1}{z-w} \right) \\ &= (1 + (z-w)Y(w) + \frac{1}{2}(z-w)^2 \partial Y(w) + \frac{1}{2}(z-w)^2 Y Y(w)) \\ &\quad \times \left( \frac{(2k+3)(k+1)}{(z-w)^3} + (k+1) \frac{R_2^+(w) + R_1^+(w) + R_0^+(w)}{(z-w)^2} \right. \\ &\quad \left. + \frac{(k+2)(\partial R_1^+ + 2\partial R_0^+) + R_2^+ R_1^+ + R_2^+ R_0^+ + R_1^+ R_0^+}{z-w} \right), \end{aligned}$$

with the rest of the calculation totally straightforward.

### 3. $\mathcal{W}_n^{(2)}$ ALGEBRAS FROM $\widehat{s\ell}(n|1)$

The second construction of the  $\mathcal{W}_n^{(2)}$  algebras is related to the cosets  $\widehat{s\ell}(n|1)/\widehat{s\ell}(n)$ , which actually becomes  $\mathcal{W}_n^{(2)}$  after a “correction” by  $e^{\pm\sqrt{n}\phi(z)}$ , where  $\phi$  is an auxiliary scalar field with the OPE (1.10). This generalizes the construction in [10] which explicitly shows that after the appropriate “correction,”  $\widehat{s\ell}(2|1)/\widehat{s\ell}(2)$  is  $\widehat{s\ell}(2)$ .

Theorem 1.2 is formulated more specifically as follows.

**3.1. Theorem.** *Let the level  $k'$  be related to  $k$  by (1.7). Let  $e_1(z), \dots, e_n(z)$  and  $f_1(z), \dots, f_n(z)$  be the two  $sl(n)$   $n$ -plets of the fermionic generators of  $\widehat{sl}(n|1)_{k'}$ . Then the operators*

$$\begin{aligned}\mathcal{E}(z) &= \frac{1}{(k' + n - 1)^{n/2}} e_1(z) e_2(z) \dots e_n(z) e^{\sqrt{n}\phi(z)}, \\ \mathcal{F}(z) &= \frac{(-1)^{n+1}}{(k' + n - 1)^{n/2}} f_1(z) f_2(z) \dots f_n(z) e^{-\sqrt{n}\phi(z)}\end{aligned}$$

*generate the  $\mathcal{W}_n^{(2)}(k)$  algebra.*

### 3.2. Outline of the proof.

**3.2.1. The second quantum group.** We begin with identifying the “second” quantum group that commutes with  $\mathcal{W}_n^{(2)}$  and with  $\mathcal{U}_q sl(n|1)$  constructed in 2.1.5 (see footnote 2 and the preceding text). For this, we define

$$\begin{aligned}S_{n,i} &= \oint e^{-\frac{1}{k+n}(a_i, \varphi)}, \quad i \neq 0, \\ S_{n,0} &= \oint (a Q_+ + b Q_-) e^{-\frac{1}{k+n}(\psi_+ + \psi_-, \varphi)} \quad (a \neq b, |a|^2 + |b|^2 \neq 0).\end{aligned}$$

A direct calculation shows the following lemma.

**3.2.2. Lemma.** *The operators  $S_{n,i}$ ,  $n - m - 1 \geq i \geq -m + 1$ , commute with the  $\mathcal{W}_{n[m]}^{(2)}$  algebra and furnish a representation of the nilpotent subalgebra of the  $\mathcal{U}_{\tilde{q}} sl(n)$  quantum group with  $\tilde{q} = e^{i\pi/(k+n)}$ .*

For  $m = 0$  (the maximally asymmetric realization), the operators in the lemma are  $S_{n,i}$ ,  $i = n - 1, \dots, 1$ . In each of the “more symmetric” realizations, which involve  $S_{n,0}$ , the arbitrariness in  $a$  and  $b$  is due to the possibility of adding a total derivative to the integrand and of choosing the overall normalization; a certain pair  $(a, b)$  is to be fixed in what follows.

**3.2.3. Remark.** In the notation  $\mathcal{U}_q sl(n|1)$  used in the previous sections,  $q$  was part of the general notation for quantum groups. We now have quantum groups with different values of the deformation parameter, and we keep the notation  $q$  for the deformation parameter of  $sl(n|1)$ ; its value is  $q = e^{i\pi(k+n)}$ , expressed through the  $k$  parameter in  $\mathcal{W}_{n[m]}^{(2)}(k)$  (for all  $m$ ). The quantum deformation of  $sl(n)$  in 3.2.2 is with the parameter  $\tilde{q} = e^{\frac{i\pi}{k+n}}$ , and we therefore use the notation  $\mathcal{U}_{\tilde{q}} sl(n)$  for the corresponding quantum group.

For each fixed  $n$  and  $m$ , we next construct vertices that are highest-weight representations of the  $\mathcal{U}_{\tilde{q}} sl(n)$  quantum group generated by the  $S_{n,i}$ . Let

$$(3.1) \quad V_{n,m}(z) = e^{(v_{n,m}(k), \varphi(z))},$$

where

$$v_{n,m}(k) = \frac{1}{n(k+n)} \left( \sum_{\substack{j=-m+1 \\ j \neq 0}}^{n-m-1} (m+j)a_j + m\psi_+ + m\psi_- \right) + \frac{1}{n}\xi.$$

The behavior of  $V_{n,m}(z)$  under  $\mathcal{U}_{\tilde{q}}sl(n)$  is described as follows. We let  $S \cdot V(z)$  denote the “dressing” of a vertex  $V(z)$  by a screening operator  $S = \oint du s(u)$ . It is given by the adjoint action of  $S$  and can be represented as the integral  $\oint du s(u)V(z)$  taken along the contour running below the real axis from minus infinity to the vicinity of  $z$ , encompassing  $z$ , and returning to minus infinity above the real axis.

**3.2.4. Lemma.** *The vertex operator  $V_{n,m}(z)$  satisfies the relations*

$$S_{n,i} \cdot V_{n,m}(z) = 0, \quad n - m - 2 \geq i \geq -m + 1$$

and

$$S_{n,n-m-1} \cdot S_{n,n-m-1} \cdot V_{n,m}(z) = 0,$$

but

$$S_{n,n-m-1} \cdot V_{n,m}(z) \neq 0,$$

where  $S_{n,i}$  are the screenings in Lemma 3.2.2. Therefore,  $V_{n,m}(z)$  generates the vector representation of  $\mathcal{U}_{\tilde{q}}sl(n)$  (the deformation of the vector representation of  $sl(n)$ ).

Let  $\mathbb{C}_q^n$  denote the  $n$ -dimensional  $\mathcal{U}_{\tilde{q}}sl(n)$ -representation generated from  $V_{n,m}(z)$ .

We next consider the properties of  $V_{n,m}(z)$  with respect to the  $\mathcal{W}_{n[m]}^{(2)}$  algebra. First, because  $(\xi, v_{n,m}(k)) = 0$ ,  $V_{n,m}(z)$  is *local* with respect to  $\mathcal{E}_{n[m]}$  and  $\mathcal{F}_{n[m]}$ . Moreover, it is actually primary with respect to  $\mathcal{E}_{n[m]}$  and  $\mathcal{F}_{n[m]}$ . To formulate this, we consider the state  $|V_{n,m}\rangle$  corresponding to  $V_{n,m}$  and introduce the modes of  $\mathcal{E}_{n[m]}$  and  $\mathcal{F}_{n[m]}$  as

$$\mathcal{E}_{n[m]}(z) = \sum_{\ell \in \mathbb{Z} + \frac{1}{2}\varepsilon_n} \mathcal{E}_{n[m],\ell} z^{-\ell - \frac{n}{2}}, \quad \mathcal{F}_{n[m]}(z) = \sum_{\ell \in \mathbb{Z} + \frac{1}{2}\varepsilon_n} \mathcal{F}_{n[m],\ell} z^{-\ell - \frac{n}{2}},$$

where  $\varepsilon_n = n \bmod 2$ , and the modes of  $\mathcal{H}$  and  $\mathcal{T}$  in the standard way, as

$$\mathcal{H}_{n[m]}(z) = \sum_{\ell \in \mathbb{Z}} \mathcal{H}_{n[m],\ell} z^{-\ell-1}, \quad \mathcal{T}_{n[m]}(z) = \sum_{\ell \in \mathbb{Z}} \mathcal{L}_{n[m],\ell} z^{-\ell-2}.$$

**3.2.5. Lemma.** *We have*

$$(3.2) \quad \begin{aligned} \mathcal{E}_{n[m],\ell} |V_{n,m}\rangle &= 0, \quad \ell \geq -\frac{n}{2} + 1, \\ \mathcal{F}_{n[m],\ell} |V_{n,m}\rangle &= 0, \quad \ell \geq -\frac{n}{2} + 2, \end{aligned}$$

and

$$(3.3) \quad \begin{aligned} \mathcal{H}_{n[m],0} |V_{n,m}\rangle &= \frac{1}{n} |V_{n,m}\rangle, \\ \mathcal{L}_{n[m],0} |V_{n,m}\rangle &= \left(1 - \frac{n}{2} + \frac{n^2 - 1}{2n(k+n)}\right) |V_{n,m}\rangle. \end{aligned}$$

For  $n = 2$ , for example, this gives the standard highest-weight conditions for  $\widehat{sl}(2)$  highest-weight vectors,  $\mathcal{E}_{\ell \geq 0} | \rangle = 0$  and  $\mathcal{F}_{\ell \geq 1} | \rangle = 0$ .

*Proof.* The annihilation conditions in the lemma are shown by directly calculating the corresponding OPEs. We recall that  $\mathcal{E}_{n[m]}(z)$  and  $\mathcal{F}_{n[m]}(z)$  involve differential polynomials of the respective degrees  $m$  and  $n - m$ ; a priori, such polynomials develop poles of the respective orders  $m$  and  $n - m$  in the OPE with a vertex operator. But the explicit (factored) form of the differential polynomials above readily implies the OPEs

$$\begin{aligned} \mathcal{E}_{n[m]}^{(k)}(z) V_{n,m}(w) &= 0, \\ \mathcal{F}_{n[m]}^{(k)}(z) V_{n,m}(w) &= \frac{\tilde{F}_{n,m}}{z - w}, \end{aligned}$$

which are equivalent to the annihilation conditions in the lemma.<sup>7</sup> The other formulas are established immediately.  $\square$

For future use, we also note that the length of the momentum of  $V_{n,m}$  is given by

$$(3.4) \quad (v_{n,m}(k), v_{n,m}(k)) = \frac{n-1}{n(k+n)}.$$

**3.2.6. A dual vertex.** Similarly to (3.1), we define the vertex operator that carries the dual vector representation of  $\mathcal{U}_{\tilde{q}sl}(n)$  and is at the same time a twisted  $\mathcal{W}_{n[m]}^{(2)}$ -primary. The results corresponding to the previous two lemmas are as follows.

For the vertex operator

$$(3.5) \quad V_{n,m}^*(z) = e^{(v_{n,m}^*(k), \varphi(z))}$$

with

$$v_{n,m}^*(k) = \frac{1}{n(k+n)} \left( \sum_{\substack{j=1 \\ j \neq n-m}}^{n-1} j a_{n-m-j} + (n-m)\psi_+ + (n-m)\psi_- \right) - \frac{1}{n}\xi,$$

it follows that

$$S_{n,i} \cdot V_{n,m}^*(z) = 0, \quad n - m - 1 \geq i \geq -m + 2$$

---

<sup>7</sup>Although we do not need this in what follows, we note that  $\tilde{F}_{n,m}(z) = :\mathcal{F}_{(n-1)[m]}^{(k+1)} V_{n,m}:(z)$ , the normal-ordered product involving the  $\mathcal{F}$  generator of the “preceding,”  $\mathcal{W}_{(n-1)[m]}^{(2)}$  algebra, which is realized in the same free-field space as discussed in Sec. 2.

and

$$S_{n,-m+1} \cdot S_{n,-m+1} \cdot V_{n,m}^*(z) = 0,$$

but

$$S_{n,-m+1} \cdot V_{n,m}^*(z) \neq 0,$$

showing that  $V_{n,m}^*(z)$  generates the dual vector representation of the quantum group  $\mathcal{U}_{\tilde{q}}sl(n)$ . We write  $\mathbb{C}_{\tilde{q}}^n$  for the  $n$ -dimensional  $\mathcal{U}_{\tilde{q}}sl(n)$ -module generated from  $V_{n,m}^*(z)$ .

Further,  $V_{n,m}^*$  is a “twisted”  $\mathcal{W}_{n[m]}^{(2)}$  primary, namely

$$(3.6) \quad \begin{aligned} \mathcal{E}_{n[m],\ell} |V_{n,m}^*\rangle &= 0, \quad \ell \geq -\frac{n}{2} + 2, \\ \mathcal{F}_{n[m],\ell} |V_{n,m}^*\rangle &= 0, \quad \ell \geq -\frac{n}{2} + 1 \end{aligned}$$

for the corresponding state. The last formulas are a reformulation of the OPEs

$$\begin{aligned} \mathcal{E}_{n[m]}^{(k)}(z) V_{n,m}^*(w) &= \frac{\tilde{E}_{n,m}}{z-w}, \\ \mathcal{F}_{n[m]}^{(k)}(z) V_{n,m}^*(w) &= 0. \end{aligned}$$

**3.2.7.  $\widehat{sl}(n)_{k'}$  vertices and quantum-group duality.** We next introduce the  $\widehat{sl}(n)$  algebra of the level  $k'$  related to  $k$  by (1.8) and the corresponding quantum group  $\mathcal{U}_{q'}sl(n)$ . The vector representation of  $\mathcal{U}_{q'}sl(n)$  can then be “coupled” with the vector representation of  $\mathcal{U}_{\tilde{q}}sl(n)$ , and similarly for the dual vector representation. Because we consider generic  $k$ , we also have generic  $k'$  determined from (1.8), and hence generic  $\tilde{q} = e^{i\pi/(k+n)}$  and  $q' = e^{i\pi/(k'+n)}$  such that  $\tilde{q}q' = -1$ .

For the  $\widehat{sl}(n)$  algebra of level  $k'$ , let  $V_{[\lambda_1,k']}(z)$  denote the vertex operator corresponding to the vector representation weight  $\lambda_1$ . We have

$$2\rho \cdot \lambda_1 = n - 1, \quad \lambda_1 \cdot \lambda_1 = 1 - \frac{1}{n},$$

where  $\rho$  is half the sum of positive roots, and hence the conformal dimension

$$\Delta_{[\lambda_1,k']} = \frac{\lambda_1 \cdot (\lambda_1 + 2\rho)}{2(k' + n)}$$

of  $V_{[\lambda_1,k']}(z)$  is given by

$$(3.7) \quad \Delta_{[\lambda_1,k']} = \frac{n^2 - 1}{2n(k' + n)}.$$

The vertex operator  $V_{[\lambda_1,k']}(z)$  has  $n$  components, which make a basis in the vector representation of  $sl(n)$  (the horizontal subalgebra in  $\widehat{sl}(n)_{k'}$ ). We use the notation  $\mathbb{C}_{\lambda_1}^n(z)$  for this space, which is the evaluation representation of  $\widehat{sl}(n)$  (with the subscript  $\lambda_1$  intended to distinguish it from other copies of  $\mathbb{C}^n$ ). In the Wakimoto bosonization [11] (also

see [12] and references therein), the highest-weight vector in  $\mathbb{C}_{\lambda_1}^n(z)$  is given by

$$V_{[\lambda_1, k']}^{(0)}(z) = e^{\frac{1}{\sqrt{k'+n}} \lambda_1 \cdot \varphi(z)}.$$

The length squared of its “momentum” is

$$(3.8) \quad \frac{\lambda_1 \cdot \lambda_1}{k' + n} = \frac{n-1}{n(k'+n)}.$$

Next,  $V_{[\lambda_1, k']}^{(0)}(z)$  carries a representation of the  $\mathcal{U}_{q'} sl(n)$  quantum group, which is the  $n$ -dimensional  $\mathcal{U}_{q'} sl(n)$ -module  $\mathbb{C}_{q'}^n$  given by the quotient of the Verma module over  $n-1$  singular vectors. The vertex  $V_{[\lambda_1, k']}^{(0)}(z)$  can therefore be represented as

$$(3.9) \quad \mathbb{C}_{\lambda_1}^n(z) \otimes \mathbb{C}_{q'}^n.$$

We also introduce the vertex operator  $V_{[\lambda_{n-1}, k']}^{(0)}(z) = V_{[\lambda_1, k']}^*(z)$  associated with the dual vector representation of  $sl(n)$ , given by

$$(3.10) \quad \mathbb{C}_{\lambda_1}^{*n}(z) \otimes \overline{\mathbb{C}}_{q'}^n,$$

with the factors dual to the respective factors in (3.9). The dimension and the momentum length squared of the lowest-weight component coincide with those in (3.7) and (3.8).

It is useful to consider the nonlocal algebra  $\mathcal{A}[sl(n)_{k'}]$  of vertex operators generated by  $\mathbb{C}_{\lambda_1}^n(z) \otimes \mathbb{C}_{q'}^n$  and  $\mathbb{C}_{\lambda_1}^{*n}(z) \otimes \overline{\mathbb{C}}_{q'}^n$  in (3.9) and (3.10). It contains  $\mathcal{U} \widehat{sl}(n)$ , and therefore carries the adjoint action of  $\mathcal{U} \widehat{sl}(n)$  (in particular, the center acts trivially); in addition, it carries an action of  $\mathcal{U}_{q'} sl(n)$ , and for generic  $k'$ ,  $\mathcal{U} \widehat{sl}(n)$  is the space of  $\mathcal{U}_{q'} sl(n)$ -invariants in  $\mathcal{A}[sl(n)_{k'}]$ .

For  $\mathcal{W}_{n[m]}^{(2)}$ , a similar nonlocal algebra  $\mathcal{A}[\mathcal{W}_{n[m]}^{(2)}(k)]$  is generated by the vertex operators

$$V_{n,m}(z) = \mathbb{C}(z) \otimes \mathbb{C}_{\tilde{q}}^n, \quad V_{n,m}^*(z) = \mathbb{C}^*(z) \otimes \overline{\mathbb{C}}_{\tilde{q}}^n.$$

constructed in 3.2.1. It also contains  $\mathcal{W}_{n[m]}^{(2)}$  (the  $\mathcal{E}_{n[m]}(z)$  and  $\mathcal{F}_{n[m]}(z)$  currents are identified in the quantum deformations of the respective products  $\bigwedge^n V_{n,m}(z)$  and  $\bigwedge^n V_{n,m}^*(z)$ ). In addition,  $\mathcal{A}[\mathcal{W}_{n[m]}^{(2)}(k)]$  carries the action of  $\mathcal{U}_{\tilde{q}} sl(n)$ , and  $\mathcal{W}_{n[m]}^{(2)}$  is the centralizer of  $\mathcal{U}_{\tilde{q}} sl(n)$  for generic  $k$ .

**3.2.8. An “almost local” subalgebra.** In  $\mathcal{A}[\mathcal{W}_{n[m]}^{(2)}(k)] \otimes \mathcal{A}[sl(n)_{k'}]$ , with  $k$  and  $k'$  related by (1.8), we now identify an “almost local” subalgebra  $\tilde{\mathcal{L}}_{n[m], k}$  by “coupling” the special subspaces in  $\mathcal{A}[\mathcal{W}_{n[m]}^{(2)}(k)]$  and  $\mathcal{A}[sl(n)_{k'}]$ . That is, we couple the  $\mathcal{W}_{n[m]}^{(2)}$  vertex operators  $V_{n,m}(z)$  and  $V_{n,m}^*(z)$  with the  $\widehat{sl}(n)_{k'}$  vertex operators  $V_{[\lambda_1, k']}^{(0)}(z)$  and  $V_{[\lambda_1, k']}^*(z)$  as follows.

Let  $R_{\tilde{q}}$  and  $R_{q'}$  be the  $R$ -matrices

$$\begin{aligned} R_{\tilde{q}} : \mathbb{C}_{\tilde{q}}^n \otimes \mathbb{C}_{\tilde{q}}^n &\rightarrow \mathbb{C}_{\tilde{q}}^n \otimes \mathbb{C}_{\tilde{q}}^n, \\ R_{q'} : \mathbb{C}_{q'}^n \otimes \mathbb{C}_{q'}^n &\rightarrow \mathbb{C}_{q'}^n \otimes \mathbb{C}_{q'}^n \end{aligned}$$

for the vector representations of  $\mathcal{U}_{\tilde{q}}sl(n)$  and  $\mathcal{U}_{q'}sl(n)$  respectively. We consider the  $R$ -matrix

$$\mathcal{R} = (R_{\tilde{q}})_{13} \otimes (R_{q'})_{24} : (\mathbb{C}_{\tilde{q}}^n \otimes \mathbb{C}_{q'}^n) \otimes (\mathbb{C}_{\tilde{q}}^n \otimes \mathbb{C}_{q'}^n) \rightarrow (\mathbb{C}_{\tilde{q}}^n \otimes \mathbb{C}_{q'}^n) \otimes (\mathbb{C}_{\tilde{q}}^n \otimes \mathbb{C}_{q'}^n)$$

and recall that (1.8) implies that  $\tilde{q}q' = -1$ . Then the tensor product  $\mathbb{C}_{\tilde{q}}^n \otimes \mathbb{C}_{q'}^n$  contains a 1-dimensional subspace  $I_n$  that is invariant under  $\mathcal{R}^2 = \mathcal{R}_{12}\mathcal{R}_{21}$  (and in fact, also under  $\mathcal{R}$ ). The eigenvalue of the thus understood  $\mathcal{R}^2$  operator on this subspace is  $e^{-\frac{2i\pi}{n}}$ , which we write as

$$(3.11) \quad \mathcal{R}^2 : I_n \otimes I_n \mapsto e^{-\frac{2i\pi}{n}} I_n \otimes I_n,$$

The dual space also contains an invariant 1-dimensional subspace,

$$(3.12) \quad \mathcal{R}^2 : I_n^* \otimes I_n^* \mapsto e^{-\frac{2i\pi}{n}} I_n^* \otimes I_n^*$$

(with  $\mathcal{R}^2$  understood appropriately), and moreover,

$$(3.13) \quad \mathcal{R}^2 : I_n \otimes I_n^* \mapsto e^{\frac{2i\pi}{n}} I_n \otimes I_n^*.$$

We next use the embedding  $I_n \hookrightarrow \mathbb{C}_{\tilde{q}}^n \otimes \mathbb{C}_{q'}^n$  to identify an  $n$ -dimensional subspace in the tensor product of the vertex operators  $V_{n,m}(z)$  and  $V_{n,m}^*(z)$ ,

$$(3.14) \quad \begin{aligned} \tilde{\mathbf{E}}(z) \equiv \mathbb{C}^n(z) \cong \mathbb{C}^n(z) \otimes I_n &\longrightarrow \underbrace{\mathbb{C}(z) \otimes \mathbb{C}_{\tilde{q}}^n}_{\parallel} \otimes \underbrace{\mathbb{C}_{\lambda_1}^n(z) \otimes \mathbb{C}_{q'}^n}_{\parallel} \\ &V_{n,m}(z) \otimes V_{[\lambda_1, k']}(z) \end{aligned}$$

and similarly with the dual spaces,

$$(3.15) \quad \begin{aligned} \tilde{\mathbf{F}}(z) \equiv \mathbb{C}^n(z) \cong \mathbb{C}^n(z) \otimes I_n &\longrightarrow \underbrace{\mathbb{C}(z) \otimes \overline{\mathbb{C}_{\tilde{q}}^n}}_{\parallel} \otimes \underbrace{\mathbb{C}_{\lambda_1}^n(z) \otimes \overline{\mathbb{C}_{q'}^n}}_{\parallel} \\ &V_{n,m}^*(z) \otimes V_{[\lambda_1, k']}^*(z) \end{aligned}$$

In view of the eigenvalues in (3.11)–(3.13), which become the monodromies of the vertex operators constituting  $\mathbb{C}^n(z)$  and  $\mathbb{C}^n(z)$ , these operators are “almost local:” the monodromies of the components of  $\tilde{\mathbf{E}}(z)$  and  $\tilde{\mathbf{F}}(z)$  with respect to each other are  $e^{\pm \frac{2i\pi}{n}}$ ; the operator products between these components are

$$\begin{aligned} \tilde{\mathbf{E}}_\alpha(z) \tilde{\mathbf{E}}_\beta(w) &= (z-w)^{-\frac{1}{n}} \mathbb{L}_{\alpha\beta}^{++}(z, w), \quad \tilde{\mathbf{F}}_\alpha(z) \tilde{\mathbf{F}}_\beta(w) = (z-w)^{-\frac{1}{n}} \mathbb{L}_{\alpha\beta}^{--}(z, w), \\ \tilde{\mathbf{E}}_\alpha(z) \tilde{\mathbf{F}}_\beta(w) &= (z-w)^{\frac{1}{n}} \mathbb{L}_{\alpha\beta}^{+-}(z, w), \end{aligned}$$

where  $\mathbb{L}_{\alpha\beta}^{**}(z, w)$  are Laurent series in  $(z - w)$ . The “almost local” subalgebra  $\tilde{\mathcal{L}}_{n[m],k}$  is the algebra generated by  $\tilde{\mathbf{E}}$  and  $\tilde{\mathbf{F}}$ .

**3.2.9. A scalar-field “correction”.** We now modify the vertex operators  $\tilde{\mathbf{E}}(z)$  and  $\tilde{\mathbf{F}}(z)$  defined in (3.14) and (3.15) to obtain fermionic currents.

We introduce an auxiliary scalar current  $\partial f$  with the operator product

$$(3.16) \quad \partial f(z) \partial f(w) = \frac{1}{(z-w)^2}$$

and consider the 1-dimensional lattice algebra  $\mathcal{F}_{\frac{1}{\sqrt{n}}}$  generated by  $e^{\pm \frac{1}{\sqrt{n}} f(z)}$ . It then follows that  $\mathcal{A}[\mathcal{W}_{n[m]}^{(2)}(k)] \otimes \mathcal{A}[sl(n)_{k'}] \otimes \mathcal{F}_{\frac{1}{\sqrt{n}}}$  contains the local subalgebra generated by the operators

$$\begin{aligned} \mathbf{E}(z) &= \tilde{\mathbf{E}}(z) e^{\frac{1}{\sqrt{n}} f(z)}, \\ \mathbf{F}(z) &= \tilde{\mathbf{F}}(z) e^{-\frac{1}{\sqrt{n}} f(z)}. \end{aligned}$$

We note that the dimension of the operators in each of the  $n$ -plets given by  $\tilde{\mathbf{E}}(z)$  and  $\tilde{\mathbf{F}}(z)$  is the sum of  $\Delta_{[\lambda_1, k']}$  in (3.7) and the dimension  $\Delta_n(k)$  of  $V_{n,m}$  read off from (3.3). Because of (1.8), we have

$$(3.17) \quad \Delta_{[\lambda_1, k']} + \Delta_n(k) = 1 - \frac{1}{2n}.$$

Therefore, the vertex operators  $\mathbf{E}(z)$  and  $\mathbf{F}(z)$  have dimension 1 (given by  $1 - \frac{1}{2n}$  in (3.17) plus  $\frac{1}{2n}$  for  $e^{\pm \frac{1}{\sqrt{n}} f}$ ). Similarly, in a bosonization where (the highest-weight component of) the vertex operator  $\mathbb{C}_{\lambda_1}^n(z)$  in the right-hand side of (3.14) is a pure exponential, the length squared of its momentum is given by adding (3.4) and (3.8),

$$(3.18) \quad \frac{n-1}{n(k+n)} + \frac{n-1}{n(k'+n)} = 1 - \frac{1}{n}.$$

Therefore, the momentum length squared of  $\mathbf{E}(z)$  and  $\mathbf{F}(z)$  in a bosonized representation is 1 (given by  $1 - \frac{1}{n}$  in (3.18) plus  $\frac{1}{n}$  for  $e^{\pm \frac{1}{\sqrt{n}} f}$ ).

To summarize,  $\mathbf{E}(z)$  and  $\mathbf{F}(z)$  are given by  $n$  fermionic dimension-1 currents each, which are local with respect to each other.  $\mathbf{E}(z)$  is in the vector representation of  $sl(n)$  and  $\mathbf{F}(z)$  is in the dual representation. *These  $2n$  dimension-1 fermionic currents are the two  $n$ -plets of the  $\widehat{sl}(n|1)$  fermionic currents*, and we thus obtain a vertex-operator extension of  $\mathcal{W}_{n[m]}^{(2)}(k) \otimes \mathcal{U}\widehat{sl}(n)_{k'}$ ,

$$\begin{array}{ccc} & \tilde{\mathcal{L}}_{n[m],k} \otimes \mathcal{F}_{\frac{1}{\sqrt{n}}} & \\ \nearrow & & \nwarrow \\ \mathcal{W}_{n[m]}^{(2)}(k) \otimes \mathcal{U}\widehat{sl}(n)_{k'} & & \mathcal{U}\widehat{sl}(n|1)_{k'} \end{array}$$



The diagram means that by “coupling” vertex operators of  $\mathcal{W}_n^{(2)}$  and  $\widehat{sl}(n)$  (and the auxiliary  $e^{\pm \frac{1}{\sqrt{n}}f}$ ), we obtain an algebra that contains  $\widehat{sl}(n|1)_{k'}$ .

This vertex-operator extension “inverts” the coset (1.9). This means, in particular, that  $\widehat{sl}(n)_{k'}$  in the left-hand part of the above diagram is the subalgebra of  $\widehat{sl}(n|1)_{k'}$  in the right, and we actually have the diagram (dropping the redundant  $[m]$ )

$$\begin{array}{ccc}
 & \tilde{\mathcal{L}}_{n,k} \otimes \mathcal{F}_{\frac{1}{\sqrt{n}}} & \\
 \nearrow & & \nwarrow \\
 \mathcal{W}_n^{(2)}(k) \otimes \mathcal{U}\widehat{sl}(n)_{k'} \otimes \mathcal{F}_{\frac{1}{\sqrt{n}}} & & \mathcal{U}\widehat{sl}(n|1)_{k'} \otimes \widehat{\mathfrak{h}}_- \\
 \uparrow & & \downarrow \\
 \mathcal{W}_n^{(2)}(k) \otimes \mathcal{U}\widehat{sl}(n)_{k'} \otimes \widehat{\mathfrak{h}}_+ & \longrightarrow & \mathcal{U}\widehat{sl}(n|1)_{k'} \otimes \mathcal{G}_{\sqrt{n}}
 \end{array}$$

The mappings and algebras are here as follows. The Heisenberg algebra  $\widehat{\mathfrak{h}}_+$  is generated by the current  $\partial f(z)$ , see (3.16). By  $e^{\pm \frac{1}{\sqrt{n}}f(z)}$ , this Heisenberg algebra is extended to the (nonlocal) lattice algebra  $\mathcal{F}_{\frac{1}{\sqrt{n}}}$ . The Heisenberg algebra  $\widehat{\mathfrak{h}}_-$  is generated by the current  $\partial \phi(z)$ , see (1.10) ( $\partial \phi(z)$  can be represented as a unique linear combination of the Cartan currents of  $\widehat{sl}(n)_{k'}$ , the  $\mathcal{H}_n$  current, and  $\partial f$  that commutes with  $\widehat{sl}(n|1)_{k'}$ ). By  $e^{\pm \sqrt{n}\phi(z)}$ , it is extended to a one-dimensional lattice vertex-operator algebra  $\mathcal{G}_{\sqrt{n}}$ . The  $\widehat{sl}(n)_{k'}$  algebra in the left-hand part is a subalgebra in  $\widehat{sl}(n|1)_{k'}$ . The algebra  $\mathcal{U}\widehat{sl}(n|1)_{k'} \otimes \mathcal{G}_{\sqrt{n}}$  contains the elements  $e_1(z) \dots e_n(z) \otimes e^{\sqrt{n}\phi(z)}$  and  $f_1(z) \dots f_n(z) \otimes e^{-\sqrt{n}\phi(z)}$  whose images in  $\tilde{\mathcal{L}}_{n,k} \otimes \mathcal{F}_{\frac{1}{\sqrt{n}}}$  coincide with the respective images of the  $\mathcal{E}(z)$  and  $\mathcal{F}(z)$  currents of  $\mathcal{W}_n^{(2)}$ ; this describes the horizontal arrow.

**3.3. Lifting the extension.** This vertex-operator extension can be viewed as resulting from a Hamiltonian reduction of another vertex-operator extension, namely of two  $\widehat{sl}(n)$  algebras with the dual levels related by (1.8). There, we take the  $\widehat{sl}(n)_k$  and  $\widehat{sl}(n)_{k'}$  vertex operators corresponding to the vector representations,  $V_{[\lambda_1, k]}$  and  $V_{[\lambda_1, k']}$ , and similarly with the dual operators. Repeating 3.2.7 for  $\widehat{sl}(n)_k$ , we view  $V_{[\lambda_1, k]}(z)$  as  $\mathbb{C}^n(z) \otimes \mathbb{C}_{\tilde{q}}^n$ ; its properties with respect to the  $\mathcal{U}_{\tilde{q}}sl(n)$  quantum group are the same as for the  $V_{n,m}$  operators of  $\mathcal{W}_{n[m]}^{(2)}$ . To distinguish between different  $\mathbb{C}^n$  spaces, we now use the notation  $\mathbb{C}_{\lambda_1}^n(z)$  for  $V_{[\lambda_1, k]}(z)$  and  $\mathbb{C}_{\lambda'_1}^n(z)$  for  $V_{[\lambda_1, k']}(z)$  (which had no prime in the previous subsection, where we had fewer  $\mathbb{C}^n$  spaces). From (3.11), we again have a 1-dimensional subspace in  $\mathbb{C}_{\tilde{q}}^n \otimes \mathbb{C}_{q'}^n$  that is invariant under the appropriately squared  $R$ -matrix (and similarly for the dual), and hence there are  $n^2$ -dimensional subspaces embedded as

$$\begin{aligned}
 \mathbb{C}^n(z) \otimes \mathbb{C}^n(z) &\longrightarrow \mathbb{C}_{\lambda_1}^n(z) \otimes \mathbb{C}_{\tilde{q}}^n \otimes \mathbb{C}_{\lambda'_1}^n(z) \otimes \mathbb{C}_{q'}^n, \\
 \mathbb{C}^n(z) \otimes \mathbb{C}^n(z) &\longrightarrow \mathbb{C}_{\lambda_1}^n(z) \otimes \overline{\mathbb{C}}_{\tilde{q}}^n \otimes \mathbb{C}_{\lambda'_1}^n(z) \otimes \overline{\mathbb{C}}_{q'}^n.
 \end{aligned}$$

The thus “coupled” vertex operators of  $\widehat{s\ell}(n)_k$  and  $\widehat{s\ell}(n)_{k'}$  generate a nonlocal algebra  $\widetilde{\mathcal{N}} \subset \mathcal{U}\widehat{s\ell}(n)_k \otimes \mathcal{U}\widehat{s\ell}(n)_{k'}$ . Similarly to  $\widetilde{\mathcal{L}}_{n,k}$ ,  $\widetilde{\mathcal{N}}$  is “almost local”; after a free-field “correction” as above, we obtain a vertex-operator algebra  $\mathcal{X}_n(k)$  with  $2n^2$  fermionic currents. Its Hamiltonian reduction then gives  $\widehat{s\ell}(n|1)$ ; under this reduction, the  $\mathbb{C}_{\lambda_1}^n(z)$  and  $\mathbb{C}_{\lambda_1}^{*n}(z)$  vertex operators of  $\widehat{s\ell}(n)$  are reduced, respectively, to the highest-weight vector  $v_+(z) \in \mathbb{C}(z)$  and the lowest-weight vector  $v_-(z) \in \mathbb{C}^n(z)$ , which are then identified with the respective  $\mathcal{W}_n^{(2)}$  vertex operators (3.1) and (3.5). We thus have

$$\begin{array}{ccccc}
 & & \widetilde{\mathcal{N}} \otimes \mathcal{F}_{\frac{1}{\sqrt{n}}} & & \\
 & \nearrow & & \nwarrow & \\
 \mathcal{U}\widehat{s\ell}(n)_k \otimes \mathcal{U}\widehat{s\ell}(n)_{k'} & & & & \mathcal{X}_n(k) \\
 \downarrow \text{Ham. red.} & & & & \downarrow \text{Ham. red.} \\
 \mathcal{W}_n^{(2)}(k) \otimes \mathcal{U}\widehat{s\ell}(n)_{k'} & \xrightarrow{\quad} & \widetilde{\mathcal{L}}_{n,k} \otimes \mathcal{F}_{\frac{1}{\sqrt{n}}} & \xleftarrow{\quad} & \widehat{s\ell}(n|1)_{k'}
 \end{array}$$

The mappings into the algebra at the top are actually into its local subalgebras. For  $n = 2$ ,  $\mathcal{X}_n(k)$  is the exceptional affine Lie superalgebra  $\widehat{D}(2|1; \alpha)$  [10], and the somewhat mysterious algebras  $\mathcal{X}_n(k)$  must therefore give its “higher” (W-)analogues.

#### 4. CONCLUSIONS

For W algebras whose defining set of screenings involves fermionic screenings, local fields can be explicitly constructed by the method used in this paper, based on the properties of quantum supergroups. Here and in [6], the simplest applications of this technique have been given; we hope that the method can be applied in much greater generality.

For the  $\mathcal{W}_n^{(2)}$  algebras constructed in this paper, it is interesting to consider integrable representations, which have many features in common with the integrable representations of  $\widehat{s\ell}(2)$ . The  $\mathcal{E}(z)$  and  $\mathcal{F}(z)$  currents then satisfy a number of relations generalizing  $\mathcal{E}(z)^{k+1} = 0$  and  $\mathcal{F}(z)^{k+1} = 0$  satisfied in the integrable  $\widehat{s\ell}(2)$  representations. The corresponding higher relations are discussed in [8]. Resolutions of the butterfly type [13] are expected to give further insight into the structure of these representations. It would also be interesting to study semi-infinite realizations of such representations, as in [14, 15], see [8, 16, 17].

The method of vertex-operator extensions is a powerful tool in constructing infinite-dimensional algebras and establishing relations between different algebras; a number of known examples are based on “duality” relations of the type of Eq. (1.8) (possibly with another integer in the right-hand side). We note that the construction of “unifying W algebras” [7], briefly recalled in the Introduction, involves an even more general relation

between the levels,  $\frac{1}{k+n} + \frac{1}{k'+m} = 1$ , and is worth being reconsidered from the standpoint of vertex-operator extensions.

### APPENDIX A. THE LOWER $\mathcal{W}_n^{(2)}$ ALGEBRAS

The basic operator product for  $\mathcal{W}_n^{(2)}$  is given by

$$\begin{aligned}
 (A.1) \quad \mathcal{E}_n(z) \mathcal{F}_n(w) &= \frac{\lambda_{n-1}(n, k)}{(z-w)^n} + \frac{n\lambda_{n-2}(n, k) \mathcal{H}_n(w)}{(z-w)^{n-1}} \\
 &+ \lambda_{n-3}(n, k) \frac{\frac{n(n-1)}{2} \mathcal{H}_n \mathcal{H}_n(w) + \frac{n((n-2)(k+n-1)-1)}{2} \partial \mathcal{H}_n(w) - (k+n) \mathcal{T}_n(w)}{(z-w)^{n-2}} \\
 &+ \left( \lambda_{n-3}(n, k) \frac{\mathcal{W}_{n,3}(w) - (k+n) \left( \frac{1}{2} \partial \mathcal{T}_{n\perp}(w) + \frac{1}{\ell_n(k)} \mathcal{H}_n \mathcal{T}_{n\perp}(w) \right)}{(z-w)^{n-3}} \right. \\
 &\quad \left. + \lambda_{n-2}(n, k) \frac{\frac{n}{6\ell_n(k)^2} \mathcal{H}_n \mathcal{H}_n \mathcal{H}_n(w) + \frac{n}{2\ell_n(k)} \partial \mathcal{H}_n \mathcal{H}_n(w) + \frac{n}{6} \partial^2 \mathcal{H}_n(w)}{(z-w)^{n-3}} \right) \\
 &\quad + \dots,
 \end{aligned}$$

where

$$(A.2) \quad \lambda_m(n, k) = \prod_{i=1}^m (i(k+n-1) - 1),$$

$\ell_n(k)$  is defined in (1.2),  $\mathcal{T}_{n\perp} = \mathcal{T}_n - \frac{1}{2\ell_n(k)} \mathcal{H}_n \mathcal{H}_n$ ,  $\mathcal{W}_{n,3}$  is a Virasoro primary dimension-3 operator with regular OPE with  $\mathcal{H}_n$ , and the dots denote lower-order poles involving operators of dimensions  $\geq 4$ . In what follows, we write the other operator products for the lower  $\mathcal{W}_n^{(2)}$  algebras and recall the realizations of these algebras obtained in accordance with **2.3–2.4**.

**A.1. The  $\mathcal{W}_1^{(2)}$  algebra.** For completeness, we note that  $\mathcal{W}_1^{(2)}$  is merely a  $\beta\gamma$  system — two *bosons* with the first-order pole in their OPE. The  $\beta\gamma$  system is well-known to be singled out as the centralizer of one fermionic screening in the two-boson system and to have two “realizations” — bosonizations with either  $\beta$  or  $\gamma$  given by the exponential times a current. In accordance with the recursion established in **2.3**, this (current) · (exponential) construct is to be encountered in the symmetric realization of the next algebra,  $\mathcal{W}_2^{(2)}$ . This is indeed the case: the symmetric realization of  $\widehat{s\ell}(2)$  can be obtained by “rebosonizing” the Wakimoto representation.

**A.2. The  $\widehat{s\ell}(2)$  algebra.** For  $n = 2$ , the  $\mathcal{W}_2^{(2)}(k) = \widehat{s\ell}(2)_k$  algebra is singled out from a three-boson system as the centralizer of either one bosonic and one fermionic or two fermionic screenings. This gives the familiar asymmetric and symmetric three-boson

realizations of  $\widehat{sl}(2)$ . In the asymmetric realization (which is  $\mathcal{W}_{2[0]}^{(2)}$  in the nomenclature in this paper), the two currents that generate the algebra are given by

$$\begin{aligned}\mathcal{E}_{2[0]} &= e^{\Xi}, \\ \mathcal{F}_{2[0]} &= -(A_1 Q + Q Q + (k+1)\partial Q) e^{-\Xi}\end{aligned}$$

(see the OPEs at the end of **2.1.3**, where we now set  $n = 2$ ). In the symmetric realization, the  $\widehat{sl}(2)$  currents are given by

$$\begin{aligned}\mathcal{E}_{2[1]} &= Q_- e^{\Xi}, \\ \mathcal{F}_{2[1]} &= Q_+ e^{-\Xi}\end{aligned}\tag{A.3}$$

(only the two open dots remain in the rigged Dynkin diagram (2.7)).

These well-known formulas can be easily obtained by directly finding the centralizer of the corresponding screenings (one fermionic and one bosonic in the first case and two fermionic in the second case).

**A.3. The BP algebra.** By definition [1, 2], the BP  $W$  algebra  $W_3^{(2)}(k)$  is a “partial” Hamiltonian reduction of the  $\widehat{sl}(3)_k$  affine algebra, also see [3]. The BP algebra is generated by the energy-momentum tensor  $\mathcal{T}$ , two bosonic spin- $\frac{3}{2}$  generators  $\mathcal{E}$  and  $\mathcal{F}$ , and a scalar current  $\mathcal{H}$ . In the operator-product form, the algebra relations are written as (with  $\mathcal{X}^+ = \mathcal{E}$  and  $\mathcal{X}^- = \mathcal{F}$  for compactness)

$$\begin{aligned}\mathcal{E}(z) \mathcal{F}(w) &= \frac{(k+1)(2k+3)}{(z-w)^3} + \frac{3(k+1)\mathcal{H}(w)}{(z-w)^2} \\ &\quad + \frac{3\mathcal{H}\mathcal{H} - (k+3)\mathcal{T} + \frac{3}{2}(k+1)\partial\mathcal{H}}{z-w}, \\ \mathcal{H}(z) \mathcal{X}^{\pm}(w) &= \frac{\pm\mathcal{X}^{\pm}}{z-w}, \quad \mathcal{T}(z) \mathcal{X}^{\pm}(w) = \frac{\frac{3}{2}\mathcal{X}^{\pm}(w)}{(z-w)^2} + \frac{\partial\mathcal{X}^{\pm}}{z-w}, \\ \mathcal{T}(z) \mathcal{H}(w) &= \frac{\mathcal{H}(w)}{(z-w)^2} + \frac{\partial\mathcal{H}}{z-w}, \\ \mathcal{T}(z) \mathcal{T}(w) &= \frac{\frac{1}{2}c_3(k)}{(z-w)^4} + \frac{2\mathcal{T}(w)}{(z-w)^2} + \frac{\partial\mathcal{T}}{z-w}, \\ \mathcal{H}(z) \mathcal{H}(w) &= \frac{\frac{1}{3}(2k+3)}{(z-w)^2}\end{aligned}$$

(with regular terms omitted in operator products and with composite operators  $(\mathcal{H}\mathcal{H})$  understood to be given by normal-ordered products).

The realizations of  $W_3^{(2)}$  are the maximally asymmetric and the “more symmetric” ones (and two more realizations obtained from these two via the automorphism exchanging  $\mathcal{E}$  and  $\mathcal{F}$ ). In the maximally asymmetric realization,

$$\mathcal{E}_{3[0]}^{(k)} = e^{\Xi}, \quad \mathcal{F}_{3[0]}^{(k)} = -\mathcal{P}_3^{(k)}(A_2, A_1, Q)e^{-\Xi},$$

with  $\mathcal{P}_3^{(k)}$  given in **2.3.7**. In the “more symmetric” realization, the generators are given by

$$\mathcal{E}_{3[1]}^{(k)} = Q_- e^\Xi, \quad \mathcal{F}_{3[1]}^{(k)} = \mathcal{P}_2^{(k+1)}(A_1, Q_+) e^{-\Xi},$$

where  $\mathcal{P}_2^{(k)}$  is given in (2.13). The OPEs between the currents involved here are specified in **2.1.5**.

#### A.4. The $\mathcal{W}_4^{(2)}$ algebra.

**A.4.1. Realizations.** The *realizations* of  $\mathcal{W}_4^{(2)}$ , described in Secs. **2.3–2.4**, are as follows. In the totally symmetric, 4[2], realization, the generators are given by

$$\mathcal{E}_{4[2]}^{(k)} = \mathcal{P}_2^{(k+2)}(A_{-1}, Q_-) e^\Xi, \quad \mathcal{F}_{4[2]}^{(k)} = -\mathcal{P}_2^{(k+2)}(A_1, Q_+) e^{-\Xi}$$

with  $\mathcal{P}_2^{(k)}(A, Q) = AQ + QQ + (k+1)\partial Q$  (see **2.3.2**). In the 4[1] realization, the generators are

$$\mathcal{E}_{4[1]}^{(k)} = Q_- e^\Xi, \quad \mathcal{F}_{4[1]}^{(k)} = \mathcal{P}_3^{(k+1)}(A_2, A_1, Q_+) e^{-\Xi},$$

where  $\mathcal{P}_3$  is given in (2.3.7). Finally, in the maximally asymmetric realization, the generators are

$$\mathcal{E}_{4[0]}^{(k)} = e^\Xi, \quad \mathcal{F}_{4[0]}^{(k)} = -\mathcal{P}_4^{(k)}(A_3, A_2, A_1, Q) e^{-\Xi},$$

where we find from **2.3.6** that

$$\begin{aligned} \mathcal{P}_4^{(k)}(A_3, A_2, A_1, Q) = & A_1 A_1 A_1 Q + 2A_1 A_1 A_2 Q + A_1 A_1 A_3 Q \\ & + 3A_1 A_1 Q Q + A_1 A_2 A_2 Q + A_1 A_2 A_3 Q + 4A_1 A_2 Q Q \\ & + 2A_1 A_3 Q Q + 3A_1 Q Q Q + A_2 A_2 Q Q + A_2 A_3 Q Q \\ & + 2A_2 Q Q Q + A_3 Q Q Q + Q Q Q Q \\ & + (k+3)(3A_1 A_1 \partial Q + 4A_1 A_2 \partial Q + 2A_1 A_3 \partial Q + A_1 \partial A_2 Q \\ & + 9A_1 \partial Q Q + A_2 A_2 \partial Q + A_2 A_3 \partial Q + 6A_2 \partial Q Q \\ & + 3A_3 \partial Q Q + 3\partial A_1 A_1 Q + 2\partial A_1 A_2 Q + \partial A_1 A_3 Q \\ & + 3\partial A_1 Q Q + \partial A_2 Q Q + 6\partial Q Q Q) \\ & + (k+3)^2(3\partial Q \partial Q + \partial A_2 \partial Q + \partial^2 A_1 Q + 3\partial A_1 \partial Q + A_3 \partial^2 Q \\ & + 2A_2 \partial^2 Q + 4\partial^2 Q Q + 3A_1 \partial^2 Q) + (k+3)^3 \partial^3 Q. \end{aligned}$$

The remaining 4[3] and 4[4] realizations follow from 4[1] and 4[0] by the automorphism exchanging  $\mathcal{E}$  and  $\mathcal{F}$ .

**A.4.2. Operator product expansions.** We now write the  $\mathcal{W}_4^{(2)}$  operator products explicitly. First, Eq. (A.1) has just 4 pole terms,

$$\begin{aligned}\mathcal{E}(z)\mathcal{F}(w) &= \frac{(k+2)(2k+5)(3k+8)}{(z-w)^4} + \frac{4(k+2)(2k+5)\mathcal{H}(w)}{(z-w)^3} \\ &\quad + (k+2)\frac{-(k+4)\mathcal{T}(w) + 6\mathcal{H}\mathcal{H}(w) + 2(2k+5)\partial\mathcal{H}}{(z-w)^2} \\ &\quad + (k+2)\left(\mathcal{W} - \frac{1}{2}(k+4)\partial\mathcal{T}(w) - \frac{4(k+4)}{3k+8}\mathcal{T}\mathcal{H}(w) + \frac{8(11k+32)}{3(3k+8)^2}\mathcal{H}\mathcal{H}\mathcal{H}(w) \right. \\ &\quad \left. + 6\partial\mathcal{H}\mathcal{H}(w) + \frac{4(26+17k+3k^2)}{3(3k+8)}\partial^2\mathcal{H}(w)\right)\frac{1}{z-w},\end{aligned}$$

where  $\mathcal{T}$  is an energy-momentum tensor with the central charge  $c_4(k)$  and  $\mathcal{W}$  is a dimension-3 Virasoro primary whose operator product with  $\mathcal{H}$  is regular. In addition to this OPE and those in (1.2), we have, writing  $\mathcal{X}^+ = \mathcal{E}$  and  $\mathcal{X}^- = \mathcal{F}$  for compactness, the operator products with  $\mathcal{W}$  given by

$$\begin{aligned}\mathcal{W}(z)\mathcal{X}^\pm(w) &= \pm\frac{2(k+4)(3k+7)(5k+16)}{(3k+8)^2}\frac{\mathcal{X}^\pm(w)}{(z-w)^3} \\ &\quad + \frac{\pm 3\frac{(k+4)(5k+16)}{2(3k+8)}\partial\mathcal{X}^\pm(w) - 6\frac{(k+4)(5k+16)}{(3k+8)^2}\mathcal{H}\mathcal{X}^\pm(w)}{(z-w)^2} \\ &\quad - \frac{k+4}{k+2}\left(\frac{8(k+3)}{3k+8}\mathcal{H}\partial\mathcal{X}^\pm + \frac{4(3k^2+15k+16)}{(3k+8)^2}\partial\mathcal{H}\mathcal{X}^\pm \mp (k+3)\partial^2\mathcal{X}^\pm \right. \\ &\quad \left. \pm \frac{2(k+4)}{3k+8}\mathcal{T}\mathcal{X}^\pm \mp \frac{4(5k+16)}{(3k+8)^2}\mathcal{H}\mathcal{H}\mathcal{X}^\pm\right)\frac{1}{z-w},\end{aligned}$$

and

$$\begin{aligned}\mathcal{W}(z)\mathcal{W}(w) &= \frac{2(k+4)(2k+5)(3k+7)(5k+16)}{3k+8}\frac{1}{(z-w)^6} \\ &\quad - \frac{(k+4)^2(5k+16)}{3k+8}\frac{3\mathcal{T}_\perp(w)}{(z-w)^4} - \frac{(k+4)^2(5k+16)}{2(3k+8)}\frac{3\partial\mathcal{T}_\perp(w)}{(z-w)^3} \\ &\quad + \left(-\frac{3(k+4)^2(5k+16)(12k^2+59k+74)}{4(3k+8)(20k^2+93k+102)}\partial^2\mathcal{T}_\perp(w) \right. \\ &\quad \left. + \frac{8(k+4)^3(5k+16)}{(3k+8)(20k^2+93k+102)}\mathcal{T}_\perp\mathcal{T}_\perp(w) + 4(k+4)\Lambda(w)\right)\frac{1}{(z-w)^2} \\ &\quad + \left(-\frac{(k+4)^2(5k+16)(12k^2+59k+74)}{6(3k+8)(20k^2+93k+102)}\partial^3\mathcal{T}_\perp \right. \\ &\quad \left. + \frac{8(k+4)^3(5k+16)}{(3k+8)(20k^2+93k+102)}\partial\mathcal{T}_\perp\mathcal{T}_\perp + 2(k+4)\partial\Lambda\right)\frac{1}{z-w},\end{aligned}$$

where  $\mathcal{T}_\perp = \mathcal{T} - \frac{2}{3k+8}\mathcal{H}\mathcal{H}$  and

$$\begin{aligned} (k+2)^2\Lambda = & \mathcal{X}^+\mathcal{X}^- - \frac{k+2}{2}\partial\mathcal{W} - \frac{4(k+2)}{3k+8}\mathcal{W}\mathcal{H} + \frac{3(k+2)^2(k+4)(6k^2+33k+46)}{2(3k+8)(20k^2+93k+102)}\partial^2\mathcal{T}_\perp \\ & - \frac{(k+2)(k+4)^2(11k+26)}{2(3k+8)(20k^2+93k+102)}\mathcal{T}_\perp\mathcal{T}_\perp + \frac{2(k+2)(k+4)}{3k+8}\partial(\mathcal{T}_\perp\mathcal{H}) \\ & + \frac{8(k+2)(k+4)}{(3k+8)^2}\mathcal{T}_\perp\mathcal{H}\mathcal{H} - \frac{8(k+2)(2k+5)}{3(3k+8)}\partial^2\mathcal{H}\mathcal{H} - \frac{2(k+2)(2k+5)}{3k+8}\partial\mathcal{H}\partial\mathcal{H} \\ & - \frac{16(k+2)(2k+5)}{(3k+8)^2}\partial\mathcal{H}\mathcal{H}\mathcal{H} - \frac{32(k+2)(2k+5)}{3(3k+8)^3}\mathcal{H}\mathcal{H}\mathcal{H}\mathcal{H} - \frac{1}{6}(k+2)(2k+5)\partial^3\mathcal{H} \end{aligned}$$

is a dimension-4 Virasoro primary field.

Next, the operator product of  $\mathcal{W}$  and  $\Lambda$  gives rise to a *single* dimension-5 Virasoro primary  $(k+2)^3\mathcal{Z} = \frac{5}{2}\mathcal{X}^+\partial\mathcal{X}^- - \frac{5}{2}\partial\mathcal{X}^+\mathcal{X}^- + \frac{2}{3k+8}\mathcal{H}\mathcal{X}^+\mathcal{X}^- + \dots$ ,

$$\begin{aligned} \mathcal{W}(z)\Lambda(w) = & -\frac{12(k+4)(3k+5)(3k+10)(4k+11)}{(k+2)(3k+8)(20k^2+93k+102)}\frac{\mathcal{W}(w)}{(z-w)^4} \\ & - \frac{4(k+4)(36k^3+279k^2+695k+550)}{(k+2)(3k+8)(20k^2+93k+102)}\frac{\partial\mathcal{W}(w)}{(z-w)^3} \\ & + (k+4)\left((k+4)\mathcal{Z}(w) + \frac{312(k+4)(3k+5)(3k+10)(4k+11)}{(k+2)(3k+8)(20k^2+93k+102)(84k^2+349k+262)}\mathcal{W}\mathcal{T}_\perp(w) \right. \\ & \left. - \frac{18(3k+5)(3k+10)(4k+11)(4k^2+29k+62)}{(k+2)(3k+8)(20k^2+93k+102)(84k^2+349k+262)}\partial^2\mathcal{W}(w)\right)\frac{1}{(z-w)^2} \\ & + (k+4)\left(\frac{2}{5}(k+4)\partial\mathcal{Z} + \frac{30(k+4)(3k+5)(3k+10)}{(k+2)^2(3k+8)(84k^2+349k+262)}\mathcal{W}\partial\mathcal{T}_\perp \right. \\ & \left. - \frac{(3k+5)(3k+10)(96k^4+1652k^3+9647k^2+22746k+18384)}{2(k+2)^2(3k+8)(20k^2+93k+102)(84k^2+349k+262)}\partial^3\mathcal{W} \right. \\ & \left. + \frac{4(k+4)(3k+5)(3k+10)(108k^2+523k+634)}{(k+2)^2(3k+8)(20k^2+93k+102)(84k^2+349k+262)}\partial\mathcal{W}\mathcal{T}_\perp\right)\frac{1}{z-w}. \end{aligned}$$

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